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EVOLUTION OF MIXED-STATE REGIONS IN TYPE-II SUPERCONDUCTORS*

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Abstract. A mean-field model for dynamics of superconducting vortices is studied. The model, consisting of an elliptic equation coupled with a hyperbolic equation with discontinuous initial data, is formulated as a system of nonlocal integrodifferential equations. We show that there exists a unique classical solution in $C^{1+\alpha}\left(\bar{\Omega}_0\right)$ for all $t > 0$, where Ω_0 is the initial vortex region that is assumed to be in $C^{1+\alpha}$. Consequently, for any time t, the vortex region Ω_t is of $C^{1+\alpha}$, and the vorticity is in $C^{\alpha}\left(\bar{\Omega}_t\right)$.

Key words. high-temperature superconductor, nonequilibrium superconductivity, mixed-state region, vorticity, London equations

AMS subject classifications. 82D55, 76C05

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1. Introduction.

One of the phenomena that characterize a superconducting material is the Meissner effect. This refers to the exclusion from the material of time-independent as well as time-varying magnetic fields.

This state of exclusion, the Meissner phase, is independent of past history. Materials are superconducting, and thus exhibit a Meissner state, only below a certain critical temperature T_c. On the other hand, at any $0 < T < T_c$, the Meissner state is destroyed and the magnetic field penetrates the whole material (normal phase) when the magnetic field exceeds some critical value $H_c(T)$. A relation between magnetic field $\mathbf{H}$ and current $\mathbf{J}$ in the material was proposed to explain the Meissner effect:

$$\lambda^2 \nabla \times \mathbf{J} + \mathbf{H} = 0, \tag{1.1}$$

where λ is a characteristic length scale. With Ampére's law,

$$\mathbf{J} = \nabla \times \mathbf{H};$$

this leads to the London equation [11]

$$\lambda^2 \nabla \times \nabla \times \mathbf{H} + \mathbf{H} = 0.$$

It follows from this equation—and this has been corroborated by experiments—that λ gives the depth of penetration of the magnetic field.

The London equations follow from the Ginzburg–Landau equations, which couple the electrodynamics to the dynamics of an order parameter, in the limit as $\kappa = \lambda/\xi$ gets arbitrarily large, where ξ, the so-called coherence length, represents the length scale on which the order parameter (density of superconductivity) varies [6]. Thus, the London equations represent a superconductor with zero stiffness in the order parameter.

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For high-κ (type-II) superconductors, the London relation (1.1) needs to be modified [11]. It was observed that the Meissner phase obtains for magnetic fields below a certain critical field H_{c_1}, that the normal phase obtains above a higher critical field H_{c_2}, and that a different phase, the so-called Abrikosov–Shubnikov, or mixed-state, phase obtains for intermediate values of the magnetic field between H_{c_1} and H_{c_2}. In this phase the magnetic field penetrates the material in the form of quantized vortices; each vortex carries a quantum of flux, ϕ_0, known as a fluxon. These vortices interact with each other and move under the influence of applied and induced currents. As $\kappa \to \infty$, the difference between H_{c_1} and H_{c_2} increases so that for high-κ materials, which include the high temperature superconductors, this mixed state is the phase of importance.

The London equations for a single vortex filament Γ are then

$$\lambda^2 \nabla \times \nabla \times \mathbf{H} + \mathbf{H} = -\phi_0 \delta_\Gamma.$$

A mean-field model for the mixed state was arrived at in [4] by averaging the above equations over the individual vortices

$$\lambda^2 \nabla \times \nabla \times \mathbf{H} + \mathbf{H} = -\boldsymbol{\omega}.$$

The variable $\boldsymbol{\omega}$ represents the density of quantum vortices and will be referred to as the vorticity. The vorticity is assumed to be convected at a velocity $\mathbf{u}$, which is the terminal speed in the presence of Lorentz forces due to the mean field (and is perpendicular to the current). In the case that the vorticity and the magnetic field remain in a fixed direction, say $x_3 - axis$, i.e., $\boldsymbol{\omega} = (0, 0, \omega)$, $\mathbf{H} = (0, 0, H)$, the complete system then reads (see [4])

$$\omega_t + \nabla \cdot (\omega u) = 0, \tag{1.2}$$

$$\Delta H - H = -\omega, \tag{1.3}$$

$$u = -sign\,(\omega)\,\nabla H. \tag{1.4}$$

In the region $\Omega_t = \{\omega(\cdot, t) \neq 0\}$, where vortices exist, the material is in the phase of the mixed state. The boundary then represents the interface between the mixed-state phase and the superconducting phase. The evolution of such a boundary is important since any such motion is manifested as electrical resistance [7]. An approach taken in [1] is to calculate the forces experienced by any vortex due to a magnetic field formed by integration over fluxons. Among other configurations, the authors studied, via numerical simulations, the evolution of the vortex lattice starting from a configuration where vortices are concentrated in a bounded region. In the mean-field setting (1.2)–(1.4), this configuration corresponds to the case of an initially isolated mixed-state domain $\bar{\Omega}_0$ evolving in the environment of a Meissner phase. In other words, the initial data should be taken as

$$\omega(x, 0) = \omega_0(x) = \varpi_0(x)\, \chi_{\bar{\Omega}_0}(x), \tag{1.5}$$

where $\chi_{\bar{\Omega}_0}$ is the characteristic function of Ω_0, which is the initial mixed-state or vortex region, and ϖ_0 is a continuous function in the whole space. In the present paper, we are mainly concerned with the problem (1.2)–(1.4) along with the initial condition (1.5).

Since the initial vorticity (1.5) is discontinuous only on $\partial\Omega_0$, one expects that the discontinuity will evolve with the velocity u. Hence, the motion equation (1.2) is understood in the distribution sense. In order to define solutions in appropriate spaces, it has been proposed in [4] to treat the system as the following free boundary problem:

$$\omega_t = \nabla \cdot (|\omega| \nabla H) \quad \text{in } \Omega_t, \tag{1.6}$$

$$\Delta H - H = -\omega \quad \text{in } \Omega_t, \tag{1.7}$$

$$\Delta H - H = 0 \quad \text{in } R^2 \backslash \Omega_t, \tag{1.8}$$

$$[H] = \left[\frac{\partial H}{\partial n}\right] = 0 \quad \text{on } \Gamma_t = \partial \Omega_t, \tag{1.9}$$

$$V_n = -sign\,(w)\,\frac{\partial H}{\partial n} \quad \text{on } \Gamma_t, \tag{1.10}$$

where Ω_t is the moving domain initially at Ω_0, n is the outward normal, V_n is the normal velocity of the moving boundary Γ_t, and the bracket $[\cdot]$ denotes the jump across Γ_t.

In the present paper we propose a different approach to deal with the problem (1.2)–(1.5). The system will be formulated as the following integro-differential equation:

$$\frac{d\Phi(x,t)}{dt} = -\int_{\Omega_t} \nabla K(\Phi(x,t) - y)\left(J(\Phi)^{-1}\varpi_0\right)\left(\Phi^{-1}(y,t)\right) dy, \tag{1.11}$$
$$\Phi(x,0) = x \quad \text{for } x \in \bar{\Omega}_0,$$

where $\Phi : \bar{\Omega}_0 \times [0,T) \mapsto R^2$, $K(x)$ is the Green's function for the elliptic equation (1.3), J is the Jacobian, and $\Phi^{-1}(\cdot,t)$ is the inverse mapping for any fixed t. One of the advantages of the above formulation is that we can work on the fixed domain.

The main intention is to investigate classical solutions Φ for the system (1.11). We shall study uniqueness, global existence, and regularity of solutions for system (1.11).

This approach is motivated by [8] in which the authors used a system analogue to (1.11) to study motion of charged particles. We shall modify the method developed in [8] to establish short-time existence and uniqueness of the solution for (1.11). The treatment for long-time existence is partially motivated by [3]. One observes that system (1.2)–(1.5) has a certain similarity to vorticity evolution for a two-dimensional incompressible Euler system. Roughly speaking, in a two-dimensional Euler system, instead of (1.3), the relationship between the vorticity and the fluid velocity is through the Biot–Savart law [10]. When $\varpi_0(x)$ is a constant (and consequently $\varpi(x,t)$ remains constant for all t), a global smooth solution for a two-dimensional Euler system was established in [3], [5]. In our system (1.2)–(1.5), the vorticity $\varpi(x,t)$ has a more complicated structure. The main idea introduced in this paper is to estimate—instead of a C^α norm as one usually did (see [3], [8], and [5])—a C^β norm of $\varpi(\cdot,t)$ for some $0 < \beta < \alpha$. We then use this norm to bound the velocity.

The paper is organized as follows. In section 2, some preliminaries, notations, and main results will be introduced. Uniqueness and short-time existence will be investigated in section 3. Section 4 will be devoted to the derivation of some a priori estimates for solutions. Global existence will be proved in section 5.

2. Preliminaries and main results. Throughout the paper, we assume that Ω_0 is a bounded domain and that

$$\partial\Omega_0 \in C^{1+\alpha}, \quad \varpi_0(x) > 0, \quad \varpi_0 \in C^\alpha. \tag{2.1}$$

Suppose that u and ω are smooth. Then the equation (1.2) can be rewritten as

$$\omega_t + (u \cdot \nabla)\,\omega = -\omega \nabla \cdot u. \tag{2.2}$$

Let $\Phi(x,t)$ be the solution of

$$\frac{d\Phi}{dt} = u(\Phi, t), \quad \Phi(x,0) = x \quad \text{for } x \in \bar{\Omega}_0. \tag{2.3}$$

By (2.2), $\omega(\Phi(x,t), t)$ solves

$$\frac{d\omega}{dt} = -\omega \nabla \cdot u. \tag{2.4}$$

Let $J(\Phi)$ be the Jacobian of Φ. It is known that $J(\Phi)$ solves

$$\frac{dJ(\Phi)}{dt} = J(\Phi)\,\nabla \cdot u \tag{2.5}$$

so that $J(\Phi)^{-1}$ is the solution of

$$\frac{dJ(\Phi)^{-1}}{dt} = -J(\Phi)^{-1}\,\nabla \cdot u. \tag{2.6}$$

Comparing (2.4) to (2.6), it follows from the uniqueness theory of ordinary differential equations (ODE) that

$$\omega(\Phi(x,t), t) = J(\Phi)^{-1}(x,t)\,\omega_0(x) \quad \text{for } x \in \bar{\Omega}_0. \tag{2.7}$$

Since at $t = 0$, $J(\Phi) = 1$, the expression (2.7) suggests that $\omega(\Phi(x,t), t) > 0$ for $x \in \bar{\Omega}_0$, provided $J(\Phi)(x,t)$ does not vanish for t. Set

$$\Omega_t = \Phi(\Omega_0, t).$$

Then Ω_t represents the mixed-state region at time t. We extend $\omega(x,t)$ by 0 for $x \notin \bar{\Omega}_t$.

Let $K(x)$ be the fundamental solution of the elliptic equation (1.3) that has the form [2]

$$K(x) = \frac{1}{2\pi} K_0(|x|) = \frac{1}{2\pi}\left(-\ln(|x|) + S(|x|)\right), \tag{2.8}$$

where K_0 is the 0th-order modified Bessel's function of the second kind (or Hankel function of imaginary part) and S is its regular part. Hence, assuming that $H(x,t) \to 0$ as $|x| \to \infty$, we have, from (1.3),

$$H(x,t) = \int_{\Omega_t} K(x-y)\,\omega(y,t)\,dy. \tag{2.9}$$

It is easy to check that

$$\nabla H(x,t) = \nabla \int_{R^2} K(x-y)\,\omega(y,t)\,dy = \int_{\Omega_t} \nabla_x K(x-y)\,\omega(y,t)\,dy. \tag{2.10}$$

Substituting this expression, (1.4), and (2.7) into (2.3), and noting that $\omega(x,t) > 0$ in $\bar{\Omega}_t$, we arrive at the following integro-differential equation for $\Phi(x,t)$ in $\bar{\Omega}_0 \times [0,T)$:

$$
(2.11) \qquad \frac{d\Phi(x,t)}{dt} = -\int_{\Omega_t} \nabla K(\Phi(x,t) - y)\left(J(\Phi)^{-1}\varpi_0\right)\left(\Phi^{-1}(y,t),t\right)dy,
$$

$$
\Phi(x,0) = x \quad \text{for } x \in \bar{\Omega}_0.
$$

Before proceeding to state our main results, we need to introduce some function spaces and notations.

For any subset $G \subseteq R^2$, multi-index $\beta = (\beta_1, \beta_2)$, $m = |\beta|$, $0 < \alpha < 1$, and any function f in G, denote by $|f|_{m+\alpha}$ and $\|f\|_{m+\alpha}$, respectively, the Hölder seminorm and norm defined as

$$
|f|_{m+\alpha} = \sup_{x,y\in G,\,|\beta|=m} \frac{\left|D^\beta f(x) - D^\beta f(y)\right|}{|x-y|^\alpha}
$$

and

$$
\|f\|_{m+\alpha} = \sup_{x\in G,\,|\beta|\le m} \left|D^\beta f(x)\right| + |f|_{m+\alpha}.
$$

Denote by $C^{m+\alpha}(G)$ the set of all functions $f(x)$ defined in G such that $\|f\|_{m+\alpha}$ is finite. If $f(x,t)$ is defined in G_t for $t < T$, where $G_t \subseteq R^2$ depends on t, we sometimes use the notation $f \in C_x^{m+\alpha}(G_t)$ to specify that $f(\cdot,t) \in C^{m+\alpha}(G_t)$ for any fixed t. For clarity, sometimes we shall also use the notations $|f(t)|_{m+\alpha,G_t}$ and $\|f(t)\|_{m+\alpha,G_t}$ to specify the dependence on the domain G_t. We also introduce the following notation:

$$
|f(t)|_{\inf,\partial G_t} = \inf_{x\in\partial G_t} |f(x,t)|.
$$

Definition 2.1. *A function $\Phi(x,t)$, defined for $(x,t) \in \bar{\Omega}_0 \times [0,T)$ with values in R^2, is called a $C^{1+\alpha}(\bar{\Omega}_0)$ solution of* (2.11) *for $t < T$ if, for any fixed $t < T$, $\Phi(\cdot,t), D_t\Phi(\cdot,t) \in C^{1+\alpha}(\bar{\Omega}_0)$, $\Phi^{-1}(\cdot,t) \in C^{1+\alpha}(\bar{\Omega}_t)$, and $\Phi(x,t)$ solves* (2.11) *pointwise in $\bar{\Omega}_0 \times [0,T)$.*

We shall verify in the next section that in the class of $C^{1+\alpha}$, formulation (2.11) is equivalent to (1.2)–(1.5). We conclude this section with a statement of the main result of this paper.

Theorem 2.2. *Assume* (2.1). *Then there exists a unique $C^{1+\alpha}(\bar{\Omega}_0)$ solution $\Phi(x,t)$ for* (2.11) *for $t > 0$. Consequently, the mixed-state region $\bar{\Omega}_t$ is of $C^{1+\alpha}$ for all $t > 0$.*

3. Short-time existence. Let $\Phi(x,t)$ be a $C^{1+\alpha}(\Omega_0)$ function for fixed t, and $J(\Phi) \neq 0$. Introduce an operator A by

$$
(3.1) \qquad A(\Phi)(x,t) = x - \int_0^t \int_{\Omega_s} \nabla K(\Phi(x,s) - z)\left(J(\Phi)^{-1}\omega_0\right)\left(\Phi^{-1}(z,s),s\right)dz\,ds.
$$

In this section, we shall show that under the assumption of (2.1), this operator has a unique fixed point in $C_x^{1+\alpha}$ for $0 < t < T$ for some $T > 0$. This fixed point is obviously a $C^{1+\alpha}(\bar{\Omega}_0)$ solution for (2.11).

Notice that at $r = 0$ the singular part of $\nabla K(x)$ (see (2.8)) is the Newtonian kernel $x/|x|$. We need the following modified version of [8, Lemma 3.1] that will be frequently used throughout the section.

LEMMA 3.1. *Let Ω be a bounded domain in R^2. Suppose that there exists a $\varphi \in C^{1+\alpha}(R^2)$ such that $\Omega = \{\varphi(x) < 0\}$ and that $\inf_{\partial\Omega} |\nabla\varphi(x)| > 0$. Define function $w(x)$ and $G(x)$, for any $g \in C^\alpha$, by*

$$w(x) = P_v \int_\Omega \nabla \left(\frac{x-z}{|x-z|^2} \right) dz,$$

$$G(x) = \int_\Omega \nabla \left(\frac{x-z}{|x-z|^2} \right) (g(x) - g(z))\, dz,$$

where P_v means the principal value. Then

$$|w|_{0,\Omega} \le c \ln(2 + \delta d(\Omega)), \tag{3.2}$$

$$|w|_{\alpha,\Omega} \le c\delta \ln(2 + \delta d(\Omega)), \tag{3.3}$$

$$|G|_{\alpha,\Omega} \le c\,|g|_{\alpha,\Omega} \ln(2 + \delta d(\Omega)), \tag{3.4}$$

where c is a constant depending only on α and Ω, $d(\Omega)$ is the diameter of Ω, and

$$\delta = \frac{|\nabla\varphi|_{\alpha,\partial\Omega}}{|\nabla\varphi|_{\inf,\partial\Omega}}. \tag{3.5}$$

Proof. By analyzing the proof of [3, Proposition 1], one found that $|w|_{0,\Omega} \le c$ if $\delta d(\Omega) < 1$. Assertion (3.2) then follows from [3, Proposition 1]. By carefully tracking various constants in the proof of [8, Lemma 3.1] and using (3.2), inequality (3.3) follows. Estimate (3.4) follows from (3.2) and [8, Lemma 3.2].

It follows that $\nabla A(\Phi)(x,t)$ exists for $x \in \Omega_0$ and that it has interior limit for $x \in \partial\Omega_0$. One can actually compute $\nabla A(\Phi)$ as

$$\nabla A(\Phi)(x,t) = I + A_1(\Phi)(x,t) + A_2(\Phi)(x,t), \tag{3.6}$$

for $x \in \bar{\Omega}_0$, where

$$A_1(\Phi)(x,t) = -\int_0^t P_v \int_{\Omega_s} \nabla^2 K(\Phi(x,s) - z) \cdot \left(J(\Phi)^{-1} \omega_0 \right) \left(\Phi^{-1}(z,s), s \right) \nabla\Phi(x,s)\, dz ds, \tag{3.7}$$

$$A_2(\Phi)(x,t) = -\frac{1}{2\pi} \int_0^t \left(J(\Phi)^{-1} \omega_0 \right)(x,s) \nabla\Phi(x,s)\, ds. \tag{3.8}$$

For convenience, in the following we will leave out the designation P_v; all singular integrals in the paper shall be understood as the principal values.

By (3.6)–(3.8), we can show that the $C^{1+\alpha}(\bar{\Omega}_0)$ solution is a weak solution in the following sense.

DEFINITION 3.2. *Assume* (2.1). *A pair of functions* $(\omega, H) \in L^2\left(R^2 \times (0,T)\right) \times L^2\left(W^{2,2}\left(R^2\right), (0,T)\right)$ *is called a weak solution of* (1.2)–(1.5) *for* $0 \le t < T$ *if*

$$\int_0^T \int_{R^2} \omega\ (x,t) \nabla H(x,t) \nabla \xi(x,t)\, dxdt$$

$$= \int_0^T \int_{R^2} \omega(x,t)\, \xi_t(x,t)\, dxdt + \int_{R^2} \omega_0(x)\, \xi(x,0)\, dx, \tag{3.9}$$

$$\Delta H(x,t) - H(x,t) = -\omega(x,t), \qquad H(x,t) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \tag{3.10}$$

for any $\xi(x,t) \in C_0^\infty\left(R^2 \times [0,T)\right)$.

PROPOSITION 3.3. *Assume* (2.1). *Let* Φ *be a* $C^{1+\alpha}\left(\bar{\Omega}_0\right)$ *solution of* (2.11) *for* $t < T$. *Define* $\omega(x,t)$ *by* (2.7) *for* $x \in \bar{\Omega}_t$ *and by* 0 *otherwise, and define* H *by* (2.9). *Then* (ω, H) *is a classical solution of* (1.2)–(1.5) *in the sense that the equations* (2.4), (1.3), *and* (1.5) *hold pointwise in* $\bar{\Omega}_t$, *with* $u = -\nabla H$ *and* $\nabla \cdot u(x,t)$ *being understood as the limit from the interior for* $x \in \partial\bar{\Omega}_t$, *and that* (1.3) *holds in* R^2. *The converse is also true. Consequently, any* $C^{1+\alpha}\left(\bar{\Omega}_0\right)$ *solution is a weak solution.*

Proof. We only sketch the proof. Suppose that Φ is a $C^{1+\alpha}\left(\bar{\Omega}_0\right)$ solution of (2.11). By (2.7), (2.11), and Lemma 3.1, it is clear that, for any $x \in \bar{\Omega}_0$, $\omega(\Phi(x,t), t)$ is differentiable in t, and that $H(\cdot,t) \in C^{2+\alpha}(\Omega_t)$. From expression (3.6)–(3.8), one can easily verify (2.4) and (1.3)–(1.5). Obviously, $(\omega, H) \in L^2 \times W^{2,2}$, and (3.10) follows from (2.9). By (2.7), for any $\xi(x,t) \in C_0^\infty\left(R^2 \times [0,T)\right)$ and $x \in \Omega_0$,

$$\frac{d}{dt}\left(\omega(\Phi(x,t),t) J(\Phi)(x,t)\, \xi(\Phi(x,t),t)\right)$$
$$= \omega(\Phi(x,t),t) J(\Phi)(x,t) \frac{d}{dt} \xi(\Phi(x,t),t)$$
$$= \omega(\Phi(x,t),t) J(\Phi)(x,t) \left(\nabla \xi(\Phi(x,t),t) \cdot \frac{d\Phi(x,t)}{dt} + \xi_t(\Phi(x,t),t)\right),$$

where $\xi_t(x,t) = \partial\xi/\partial t$. Hence, by integration in x and t, we obtain, by (2.3),

$$-\int_{\Omega_0} \omega_0(x)\, \xi(x,0)\, dx = -\int_0^T \int_{\Omega_0} \omega(\Phi(x,t),t) \nabla H \cdot \nabla \xi(\Phi(x,t),t) J(\Phi)(x,t)\, dxdt$$
$$+ \int_0^T \int_{\Omega_0} \omega(\Phi(x,t),t)\, \xi_t(\Phi(x,t),t) J(\Phi)(x,t)\, dxdt.$$

Assertion (3.9) follows by changes of variables $y = \Phi(x,t)$ and by the fact that $\omega(x,t) = 0$ outside $\bar{\Omega}_t$.

We next derive some $C^{1+\alpha}$ estimates for the operator A defined in (3.1) and use them to establish a fixed point. Notice that the function K_0 in (2.8) has the specific form

$$K_0(r) = -\ln r + S(r), \qquad S(r) = -\left(\ln\frac{r}{2} + \gamma\right) I_0 - \ln\frac{\gamma}{2} + I_1,$$

where γ is the Euler constant (≈ 0.56), and

$$I_0 = \sum_{i=1}^{\infty} \frac{(r/2)^{2i}}{i!\Gamma(i+1)}, \quad I_1 = \sum_{i=1}^{\infty} \frac{(r/2)^{2i}}{(i!)^2} \left(\sum_{j=1}^{i} \frac{1}{j} \right).$$

The following properties can be verified through direct computations [2]:

(i) $S(r)$ is smooth for $r \geq 0$;

(ii) $K_0(r) = (e^{-r}/\sqrt{r})(1 + O((1/r)))$, as $r \to \infty$;

(iii) $K_0(r) = -\ln r + O(r)$, as $r \to 0$.

We need the following estimates for solutions of the equation (1.3).

LEMMA 3.4. *Suppose that* $\omega(x) = \varpi(x)\chi_\Omega$, *where* $\varpi \in C^\alpha$, $\Omega = \{\varphi(x) < 0\}$ *with* $\varphi \in C^{1+\alpha}$, $|\nabla\varphi|_{\inf,\partial\Omega} > 0$. *Let* H *be the solution of* (1.3) *that vanishes at infinity. Then*

$$0 \leq H \leq |\varpi|_{0,\Omega}, \tag{3.11}$$

$$\|H\|_{2,\Omega} \leq c\,|\varpi|_{0,\Omega} \ln\left[(2 + \delta d(\Omega))\left(2 + |\varpi|_{0,\Omega}^{-1}\,|\varpi|_{\alpha,\Omega}\, d(\Omega)\right)\right], \tag{3.12}$$

$$\|H\|_{2+\alpha,\Omega} \leq c\left(\|\varpi\|_{\alpha,\Omega} + \delta\,|\varpi|_{0,\Omega}\right)\ln(2 + \delta d(\Omega)), \tag{3.13}$$

where c *is a universal constant,* $d(\Omega)$ *is the diameter of* Ω, *and*

$$\delta = \frac{|\nabla\varphi|_{\alpha,\partial\Omega}}{|\nabla\varphi|_{\inf,\partial\Omega}}.$$

Proof. The inequalities (3.11) follow from the maximum principle for elliptic equations. By (2.10), for $x \in \bar{\Omega}$, we have the integral formula for $\nabla^2 H$ (analogous to (3.6)) as follows:

$$\begin{aligned}\nabla^2 H(x,t) &= \int_\Omega \nabla^2 K(x-z)\,\omega(z)\,dz + \frac{1}{2\pi}\omega(x)\\ &= \frac{1}{2\pi}\int_\Omega \nabla\left(\frac{x-z}{|x-z|^2}\right) dz\omega(x)\\ &\quad+\frac{1}{2\pi}\int_\Omega \nabla\left(\frac{x-z}{|x-z|^2}\right)(\omega(z)-\omega(x))\,dz\\ &\quad+\frac{1}{2\pi}\omega(x) + \int_\Omega \nabla^2 S(|x-z|)\,\omega(z)\,dz = k_1 + k_2 + k_3 + k_4.\end{aligned} \tag{3.14}$$

By Lemma 3.1, it is easy to see that

$$\|k_1\|_\alpha + \|k_3\|_\alpha \leq c\left(\|\varpi\|_{\alpha,\Omega} + \delta\,|\varpi|_{0,\Omega}\right)\ln(2 + \delta d(\Omega)). \tag{3.15}$$

To estimate k_2, we write

$$k_2 = \frac{1}{2\pi}\left(\int_{\Omega\setminus B_\varepsilon} + \int_{B_\varepsilon}\right)\nabla\left(\frac{x-z}{|x-z|^2}\right)(\omega(z)-\omega(x))\,dz = k_{21} + k_{22},$$

where B_ε is the ball centered at x with radius ε that will be chosen later on. By integration,

$$|k_{21}| \leq c\,|\varpi|_{0,\Omega} \int\limits_{\Omega\setminus B_\varepsilon} \frac{1}{|x-z|^2}\,dz \leq c\,|\varpi|_{0,\Omega} \ln\left(\frac{d(\Omega)}{\varepsilon}\right)$$

if $\varepsilon \leq d(\Omega)$. Otherwise, $k_{21} = 0$. Since $\varpi \in C^\alpha$, we find that

$$|k_{22}| \leq c\,|\varpi|_{\alpha,\Omega} \int\limits_{B_\varepsilon} \frac{1}{|x-z|^{2-\alpha}}\,dz \leq c\,|\varpi|_{\alpha,\Omega}\,\varepsilon^\alpha.$$

By choosing $\varepsilon^\alpha = |\varpi|_{0,\Omega}\left(|\varpi|_{\alpha,\Omega}\right)^{-1}$, we deduce that

$$|k_2| \leq c\,|\varpi|_{0,\Omega}\left(1 + \left|\ln\left(2 + |\varpi|_{0,\Omega}^{-1}\,|\varpi|_{\alpha,\Omega}\,d(\Omega)\right)\right|\right). \tag{3.16}$$

By Lemma 3.1, we also have

$$|k_2|_\alpha \leq c\,|\varpi|_{\alpha,\Omega} \ln(2 + \delta d(\Omega)). \tag{3.17}$$

Combining (3.15)–(3.17), one sees that $|k_1 + k_2 + k_3|_0$ is bounded by the left-hand side of (3.12) and that $\|k_1 + k_2 + k_3\|_\alpha$ is bounded by the left-hand side of (3.13). The assertions thus follow from the fact that $\nabla^2 S$ is smooth in R^2 and that $\nabla^2 S$ decays at the rate r^{-2} as $r \to \infty$.

THEOREM 3.5. *Suppose that the initial data satisfy* (2.1). *Then there exists a unique* $C^{1+\alpha}(\Omega_0)$ *solution* $\Phi(x,t)$ *of* (2.11) *for* $t < T$ *for some* $T > 0$.

Proof. For simplicity, we assume that there exists $\varphi_0 \in C^{1+\alpha}$ with $|\nabla\varphi_0|_{\inf,\partial\Omega_0} \neq 0$ such that $\Omega_0 = \{\varphi_0(x) < 0\}$. For any $M, T > 0$ to be chosen later on, we define a set $W(M,T)$ of vector value functions in $\bar{\Omega}_0 \times [0,T)$ as follows:

$$\begin{aligned} W(M,T) = \{&\Phi(x,t) \in R^2 :\ \Phi(x,0) = x,\\ &\|\Phi(t)\|_{1+\alpha,\Omega_0} \leq M,\ \|\Phi(x)\|_{\alpha,[0,T)} \leq M,\ \ |\nabla\Phi - I|_0 \leq 1/2\}. \end{aligned}$$

Since $|\nabla\Phi - I| \leq 1/2$, $\Phi^{-1}(\cdot,t)$ exists and maps Ω_t onto Ω_0 so that the mapping A defined in (3.1) is well defined for any $\Phi \in W(M,T)$. By applying Lemma 3.4 to $A(\Phi)$ with

$$\varpi = \left(J(\Phi)^{-1}\omega_0\right)\left(\Phi^{-1}(x,s),s\right), \tag{3.18}$$

we find that

$$\begin{aligned} \|A(\Phi)\|_{1+\alpha,\Omega_0} \leq c_0 + c\int_0^t &\left(\|\Phi\|_{1+\alpha,\Omega_0}\right)^{1+\alpha}\left(1 + \delta_s\,|\varpi|_{0,\Omega_s} + \|\varpi\|_{\alpha,\Omega_s}\right)\\ &\cdot\left(\ln(2 + d(\Omega_s)\,\delta_s) + \ln\left(2 + |\varpi|_{0,\Omega}^{-1}\,\|\varpi\|_{\alpha,\Omega_s}\,d(\Omega_s)\right)\right)ds, \end{aligned} \tag{3.19}$$

where c is a constant independent of Φ, $c_0 = \sup_{x\in\Omega_0}|x|$, δ_t is defined in terms of Ω_t by

$$\delta_t = \frac{|\nabla\varphi(x,t)|_{\alpha,\partial\Omega_t}}{|\nabla\varphi(x,t)|_{\inf,\partial\Omega_t}} = \frac{\left|\nabla\varphi_0\left(\Phi^{-1}(x,t)\right)\nabla\Phi^{-1}(x,t)\right|_{\alpha,\partial\Omega_t}}{|\nabla\varphi_0(\Phi^{-1}(x,t))\,\nabla\Phi^{-1}(x,t)|_{\inf,\partial\Omega_t}}, \tag{3.20}$$

since $\Omega_t = \{\varphi(x,t) < 0\}$, where $\varphi(x,t) = \varphi_0\left(\Phi^{-1}(x,t)\right)$. Notice that, by direct calculation,

$$\left|\nabla\Phi^{-1}\right|_{\alpha,\Omega_t} \le |\nabla\Phi|_{\alpha,\Omega_0}\left(\left|\nabla\Phi^{-1}\right|_{0,\Omega_t}\right)^{2+\alpha},$$
$$\left\|J(\Phi)^{-1}\right\|_{\alpha,\Omega_t} \le c_0\left(|\nabla\Phi|_{\alpha,\Omega_0}+1\right)\left(\left|\nabla\Phi^{-1}\right|_{0,\Omega_t}+1\right)^{3+\alpha},$$

where c_0 is a universal constant. Since $\Phi \in W(M,T)$, it is easy to see that

$$\delta_t \le c(M), \quad d(\Omega_t) \le 2|\Phi|_0 \le 2M, \quad \|\varpi\|_{\alpha,\Omega_t} \le c(M),$$

where the last inequality is due to (3.18), and the constant $c(M)$ is a polynomial of M with coefficients depending only on initial data. Hence (3.19) results in, for $t < T$,

$$\|A(\Phi)(t)\|_{1+\alpha,\Omega_0} \le c_0 + c(M)T.$$

It is easy to derive the following estimates:

$$\|A(\Phi)(x,\cdot)\|_\alpha \le c(M)T^{1-\alpha}$$

and

$$|\nabla(A(\Phi)) - I| \le c(M)T.$$

We now choose $M = 1+c_0$ and $T = \min\left(\left(2C(M)\right)^{-1}, (1+c_0)^{1/(1-\alpha)}C(M)^{-1/(1-\alpha)}\right)$. Then $A(\Phi) \in W(M,T)$. The mapping A maps $W(M,T)$ into itself.

For any $\Phi,\ \tilde{\Phi} \in W(M,T)$. Set $\Omega_t = \Phi(\Omega_0,t)$, $\tilde{\Omega}_t = \tilde{\Phi}(\Omega_0,t)$ and define

$$\rho(t) = \left|\Phi(t) - \tilde{\Phi}(t)\right|_{0,\Omega_0}.$$

By changing variables in the expressions for $A(\Phi)$ and $A\left(\tilde{\Phi}\right)$, we obtain, from (3.1),

$$A(\Phi)(x,t) = x + \int_0^t\int_{\Omega_0}\nabla K(\Phi(x,s)-\Phi(z,s))\,\omega_0(z)\,dzds,$$
$$A\left(\tilde{\Phi}\right)(x,t) = x + \int_0^t\int_{\Omega_0}\nabla K\left(\tilde{\Phi}(x,s)-\tilde{\Phi}(z,s)\right)\omega_0(z)\,dzds.$$

It follows that

$$\begin{aligned}&\left|A(\Phi)(x,t) - A\left(\tilde{\Phi}\right)(x,t)\right|\\ &\le c\int_0^t\int_{\Omega_0}\left|\nabla K(\Phi(x,s)-\Phi(z,s)) - \nabla K\left(\tilde{\Phi}(x,s)-\tilde{\Phi}(z,s)\right)\right|dzds. \qquad (3.21)\end{aligned}$$

For $\varepsilon > 0$ to be determined later, we decompose the right-hand side of (3.21) as

$$\begin{aligned}&\int_{\Omega_0}\left|\nabla K(\Phi(x,s)-\Phi(z,s)) - \nabla K\left(\tilde{\Phi}(x,s)-\tilde{\Phi}(z,s)\right)\right|dz \qquad (3.22)\\ &\qquad = \int_{\Omega_0\setminus B_\varepsilon(x)} + \int_{\Omega_0\cap B_\varepsilon(x)} = k_1 + k_2.\end{aligned}$$

From (2.8),

$$\nabla K(z) = -\frac{z}{2\pi |z|^2} + S'(|z|)\frac{z}{2\pi |z|}.$$

Since $\nabla\Phi^{-1}$ and $\nabla\tilde{\Phi}^{-1}$ are bounded, we have

$$|\Phi(x,s) - \Phi(z,s)|, \qquad \left|\tilde{\Phi}(x,s) - \tilde{\Phi}(z,s)\right| \geq c\,|x-z|.$$

Therefore,

$$\begin{aligned}&\left|\nabla K(\Phi(x,s) - \Phi(z,s)) - \nabla K\left(\tilde{\Phi}(x,s) - \tilde{\Phi}(z,s)\right)\right| \\ &\quad \leq c\rho(s)\left(1 + \frac{1}{|x-z|} + \frac{1}{|x-z|^2}\right).\end{aligned}$$

Consequently, k_1 in (3.22) is bounded by

$$|k_1| \leq c\rho(s)\int_{\varepsilon}^{d(\Omega_0)}\left(r + 1 + \frac{1}{r}\right)dr = c\rho(s)(1 + |\ln\varepsilon|). \tag{3.23}$$

Since $|\Phi(x,s) - \Phi(z,s)| \leq c\varepsilon$ for $|x-z| \leq \varepsilon$, we know that

$$S'(|\Phi(x,s) - \Phi(z,s)|) \leq c\varepsilon \quad \text{for} \quad |x-z| \leq \varepsilon.$$

It follows from the obvious estimates

$$\begin{aligned}|\nabla K(\Phi(x,t) - \Phi(z,t))| &\leq |\Phi(x,t) - \Phi(z,t)|^{-1} + S'(|\Phi(x,s) - \Phi(z,s)|) \\ &\leq c\left(1 + |x-z|^{-1}\right)\end{aligned}$$

and

$$\begin{aligned}\left|\nabla K\left(\tilde{\Phi}(x,s) - \tilde{\Phi}(z,s)\right)\right| &\leq \left|\tilde{\Phi}(x,s) - \tilde{\Phi}(z,s)\right|^{-1} + S'(|\Phi(x,s) - \Phi(z,s)|) \\ &\leq c\left(1 + |x-z|^{-1}\right)\end{aligned}$$

that the term k_2 in (3.22) can be estimated as

$$|k_2| \leq c\int_0^{\varepsilon}\left(1 + \frac{1}{r}\right)r\,dr \leq c\left(\varepsilon + \varepsilon^2\right). \tag{3.24}$$

We now choose $\varepsilon = \min(\rho(t), 1)$. From (3.21)–(3.24), it follows that

$$\left|A(\Phi)(x,t) - A\left(\tilde{\Phi}\right)(x,t)\right| \leq c\int_0^t \rho(t)(1 + |\ln\rho(t)|)\,dt. \tag{3.25}$$

Define a sequence $\Phi_n(x,t)$ by

$$\Phi_0 = x, \qquad \Phi_{n+1}(x,t) = A(\Phi_n)(x,t).$$

Since $\Phi_n \in W(M,T)$, this sequence $\{\Phi_n\}$ is precompact under the $C^{1+\gamma,\gamma}_{x,t}$ norm for any $\gamma < \alpha$. Hence we can select a subsequence, still denoting it as $\{\Phi_n\}$, and a function $\Phi \in C^{1+\gamma,\gamma}_{x,t}(\Omega_0 \times [0,T))$ such that

$$\Phi_n \longrightarrow \Phi \text{ in } C^{1+\gamma,\gamma}_{x,t}\text{norm}.$$

This implies that $\Phi \in W(M,T)$ (by checking from the definitions) and by (3.25),

$$\begin{aligned}|\Phi_{n+1}(x,t) - A(\Phi)(x,t)| &= |A(\Phi_n)(x,t) - A(\Phi)(x,t)| \\ &\le cT \sup_{0\le t\le T}\left(|\Phi_n(t) - \Phi(t)|_{0,\Omega_0}\left|\ln|\Phi_n(t) - \Phi(t)|_{0,\Omega_0}\right|\right).\end{aligned}$$

Letting $n \to \infty$, we find that Φ is a fixed point for A, i.e., $A(\Phi) = \Phi$. Next, set

$$\rho_n(t) = \sup_x |\Phi_n(x,t) - \Phi(x,t)|.$$

It follows from (3.25) that

$$\rho_{n+1}(t) \le c\int_0^t \rho_n(t)\,(1 + |\ln\rho_n(t)|)\,dt.$$

By [9, section 9], it follows that

$$\rho_n(t) \le c\,|T\ln T|^n.$$

For small T, the above inequality implies uniqueness of the fixed point. The proof is complete.

Using exactly the same argument, we can extend the assertions of Theorem 3.5 to the equation (2.11) with more general initial data:

$$\text{(3.26)}\qquad \begin{aligned}&\frac{d\Phi(x,t)}{dt} = -\int_{\Phi(\bar\Omega_0,t)} \nabla K(\Phi(x,t) - y)\left(J(\Phi)^{-1}\varpi_0\right)\left(\Phi^{-1}(y,t)\right)dy,\\ &\Phi(x,0) = \Phi_0(x) \quad \text{for } x \in \bar\Omega_0.\end{aligned}$$

COROLLARY 3.6. *In addition to the assumptions in Theorem* 3.5*, suppose also that* $\Phi_0 \in C^{1+\alpha}(\Omega_0)$, $J(\Phi_0) \ne 0$. *Then there exists a unique* $C^{1+\alpha}(\Omega_0)$ *solution* $\Phi(x,t)$ *for* (3.26) *for* $t < T$ *for some* $T > 0$. *Moreover,* T *depends only on* $\|\Phi_0\|_{1+\alpha}$, $\left\|\Phi_0^{-1}\right\|_{1+\alpha}$, $d(\Omega_0)$ *and* δ_0.

Corollary 3.6 will be used in the next section to extend the solutions for large time.

4. A priori estimates. From Corollary 3.6, it appears that a priori estimates on the $C^{1+\alpha}$ norms of Φ and Φ^{-1} will be sufficient to establish existence of global solutions. We first show that, actually, a uniform bound on the vorticity ω will be enough to guarantee that the solution can be extended for all $t > 0$.

LEMMA 4.1. *Suppose that* $\Phi(x,t)$ *is the* $C^{1+\alpha}$ *solution. Then, for* $x \in \bar\Omega_0$,

$$\text{(4.1)}\qquad \omega(\Phi(x,t),t) = \frac{\omega_0(x)\,e^{\sigma(x,t)}}{1 + \omega_0(x)\int_0^t e^{\sigma(x,s)}ds},$$

where

$$\sigma(x,t) = \int_0^t H(\Phi(x,s), s)\, ds. \tag{4.2}$$

Proof. From (2.4), (1.3), (1.4), and the fact that $\omega(\Phi(x,t), t) > 0$ for $x \in \bar{\Omega}_0$, one sees that $\omega(\Phi(x,t), t)$ is the solution of the following ODE:

$$\frac{d\omega}{dt} = \omega H - \omega^2. \tag{4.3}$$

Let $p(t) = \omega(\Phi(x,t), t)^{-1}$. Then p solves

$$\frac{dp}{dt} = -pH + 1.$$

Integrating this ODE directly, we obtain (4.1).

LEMMA 4.2. *Let $\Phi(x,t)$ be the $C^{1+\alpha}$ solution for $t < T$. Suppose that*

$$\eta(T) \equiv \int_0^T \|\varpi(t)\|_{0,\Omega_t}\, dt < \infty. \tag{4.4}$$

Then there exists a $0 < \beta(T) \le \alpha$ depending only on $\eta(T)$ such that for $t < T$,

$$\|\varpi(t)\|_{\beta,\Omega_t} \le c(\eta(T)), \tag{4.5}$$

$c(\eta(T))$ being a constant depending only on $\eta(T)$ and T.

Proof. From Lemma 3.4, we know that $u = -\nabla H$ is Lipschitz in Ω_t. However, the Lipschitz constant may depend on the C^α characters of ω and the domain Ω_t. We claim that ∇H is quasi-Lipschitz, with the constant depending only on $\|\varpi(t)\|_{0,\Omega_t}$, i.e., for $x, y \in \bar{\Omega}_t$, $|x-y| \le 1/2$,

$$|\nabla H(x,t) - \nabla H(y,t)| \le c\,\|\varpi(t)\|_{0,\Omega_t} |x-y| (1 + |\ln|x-y||), \tag{4.6}$$

where c is a universal constant. We point out at this moment that c in general also depends on the $L^1(\bar{\Omega}_t)$ norm of $\varpi(\cdot,t)$. However, by (2.7), our previous claim remains true.

Indeed, from (2.8) and (2.9), we have

$$\begin{aligned}\nabla H(x,t) &= \frac{1}{2\pi}\int_{\Omega_t} \frac{x-z}{|x-z|^2}\varpi(z,t)\, dz + \int_{\Omega_t} \nabla_x S(|x-z|)\,\varpi(z,t)\, dz \\ &= \nabla H_1(x,t) + \nabla H_2(x,t).\end{aligned}$$

The results in [10] lead to

$$|\nabla H_1(x,t) - \nabla H_1(y,t)| \le c\,\|\varpi(t)\|_{0,\Omega_t} |x-y| (1 + |\ln|x-y||).$$

Since ∇S is smooth and $S'(r) \sim r^{-1}$ for large r, the inequality (4.6) thus follows immediately.

Now, since

$$\frac{d\Phi}{dt} = -\nabla H(\Phi, t),$$

it follows that the quantity $\rho(t) = |\Phi(x,t) - \Phi(y,t)|^2$, for $x \neq y \in \bar{\Omega}_t$, satisfies

$$\left|\frac{d\rho(t)}{dt}\right| \leq c\,\|\varpi(t)\|_{0,\Omega_t}\,\rho(t)\,(1 + |\ln \rho(t)|).$$

By the Gronwall lemma, we deduce that

$$c\,|x-y|^{1/\tilde{\beta}(t)} \leq \rho(t) = |\Phi(x,t) - \Phi(y,t)| \leq c\,|x-y|^{\tilde{\beta}(t)}, \tag{4.7}$$

where $\tilde{\beta}(t) = \exp(-c\eta(t))$ is decreasing in t. The inequality (4.7) implies

$$|\Phi(t)|_{\tilde{\beta}(t),\Omega_0} + \left|\Phi^{-1}(t)\right|_{\tilde{\beta}(t),\Omega_t} \leq c.$$

By the regularity theory for elliptic partial differential equations (PDEs), we know that the solution H of (1.3) is of the class C^1 and

$$\|\nabla H(t)\|_0 \leq c\,\|\varpi(t)\|_{0,\Omega_t}. \tag{4.8}$$

Hence, the function $\sigma(x,t)$ defined in (4.2) is of the class C^{β} and

$$|\sigma(t)|_{\tilde{\beta}(t),\Omega_0} \leq \int_0^t \|\nabla H(s)\|_0\,|\Phi(s)|_{\tilde{\beta}(s),\Omega_0}\,ds \leq c\eta(t),$$

for $t < T$. The bounds on $|\sigma(t)|_{0,\Omega_0}$ can be easily derived from (4.2). Choose $\beta(T) = \min(\tilde{\beta}(T)^2, \alpha)$. It then easy to see that the assertion (4.5) hence follows from (4.1).

LEMMA 4.3. *Under the assumptions of Lemma* 4.1*, we have*

$$\left|\nabla\left(\Phi\left(\Phi^{-1}(x,t), s\right)\right)\right| \leq \exp\left(\int_s^t |\nabla u(\tau)|_0\,d\tau\right),$$

for $0 < s < t$*, and*

$$|\nabla\varphi(t)|_{\inf,\partial\Omega_t} \geq |\nabla\varphi(0)|_{\inf,\partial\Omega_0} \exp\left(-\int_0^t |\nabla u(\tau)|_0\,d\tau\right).$$

Proof. The first inequality follows from the dynamical property of (2.3). Since $\varphi(\Phi(x,t),t) = \varphi_0(x)$, we have

$$\nabla\varphi(\Phi(x,t),t) = \left((\nabla\Phi(x,t))^T\right)^{-1} \nabla\varphi_0(x).$$

By (2.3), we have

$$\frac{d}{dt}(\nabla\Phi(x,t))^{-1} = -(\nabla\Phi(x,t))^{-1}\nabla u(\Phi(x,t),t) = (\nabla\Phi(x,t))^{-1}\nabla^2 H(\Phi(x,t),t).$$

Hence,

$$\frac{d}{dt}\nabla\varphi\left(\Phi\left(x,t\right),t\right)=\left(\nabla^{2}H\right)^{T}\nabla\varphi\left(\Phi\left(x,t\right),t\right).$$

The second assertion of Lemma 4.3 follows from the Gronwall lemma.

LEMMA 4.4. *Under the assumptions of Lemma* 4.1, *we have*

$$\delta_{t,\beta}\leq c\left(\eta\left(T\right)\right),$$

where $\delta_{t,\beta}$ *is defined in* (3.20) *with* α *being replaced with* $\beta=\beta\left(T\right)$.

Proof. We recall that by definition, $\Omega_t=\{\varphi<0\}$ with $\varphi=\varphi_0\left(\Phi^{-1}\left(x,t\right)\right)$. Since Φ solves (2.3), we find that φ satisfies

$$\frac{\partial\varphi}{\partial t}+u\cdot\nabla\varphi=0,\qquad u=-\nabla H.$$

Hence $\nabla\varphi$ satisfies

$$\frac{\partial\nabla\varphi}{\partial t}+\left(u\cdot\nabla\right)\nabla\varphi=-\left(\nabla u\right)^{\top}\nabla\varphi. \tag{4.9}$$

Since $\nabla\cdot u=-\Delta H=H-\omega$, this equation can be rewritten as

$$\frac{\partial\nabla^{\perp}\varphi}{\partial t}+\left(u\cdot\nabla\right)\nabla^{\perp}\varphi=\nabla u\nabla^{\perp}\varphi+\left(\omega-H\right)\nabla^{\perp}\varphi, \tag{4.10}$$

where $\nabla^{\perp}\varphi=\left(-D_2\varphi,D_1\varphi\right)$ is divergence free and tangential to $\partial\Omega_t$. By (3.14), we have

$$\begin{aligned}\nabla u&=-\nabla^{2}H\left(x,t\right)\\&=-\frac{1}{2\pi}\int_{\Omega_t}\nabla\left(\frac{x-z}{\left|x-z\right|^{2}}\right)dz\omega\left(x,t\right)\\&\quad-\frac{1}{2\pi}\int_{\Omega_t}\nabla\left(\frac{x-z}{\left|x-z\right|^{2}}\right)\left(\omega\left(z,t\right)-\omega\left(x,t\right)\right)dz\\&\quad-\frac{1}{2\pi}\omega\left(x,t\right)-\int_{\Omega_t}\nabla^{2}S\left(\left|x-z\right|\right)\omega\left(z,t\right)dz=k_1+k_2+k_3+k_4.\end{aligned}$$

Since $\nabla^{\perp}\varphi$ is divergence free and tangential to $\partial\Omega_t$, we find by direct computation that

$$k_1\nabla^{\perp}\varphi=\int_{\Omega_t}\nabla\left(\frac{x-z}{\left|x-z\right|^{2}}\right)\left(\nabla^{\perp}\varphi\left(x,t\right)-\nabla^{\perp}\varphi\left(z,t\right)\right)dz\omega\left(x,t\right).$$

Consequently we deduce that, by Lemma 3.1, for $t\leq T$,

$$\begin{aligned}\left\|k_1\nabla^{\perp}\varphi\left(t\right)\right\|_{\beta}&\leq c\left(\left|\omega\left(t\right)\right|_{\beta}\left|\nabla^{\perp}\varphi\left(t\right)\right|_{0}+\left|\omega\right|_{0}\left|\nabla^{\perp}\varphi\left(t\right)\right|_{\beta}\right)\\&\qquad\cdot\left(\ln\left(2+\left|\omega\right|_{0}^{-1}\left|\omega\left(t\right)\right|_{\beta}d\left(\Omega_t\right)\right)+\ln\left(2+d\left(\Omega_t\right)\delta_{t,\beta}\right)\right),\end{aligned}$$

where $\beta = \beta(T)$ obtained in Lemma 4.2. In a similar manner, we have

$$\left\|k_2 \nabla^{\perp} \varphi(t)\right\|_{\beta} \le c\left(|\omega(t)|_{\beta}\left|\nabla^{\perp} \varphi(t)\right|_0 + |\omega|_0\left|\nabla^{\perp} \varphi(t)\right|_{\beta}\right) \cdot\left(\ln\left(2+|\omega|_0^{-1}|\omega(t)|_{\beta}\, d(\Omega_t)\right)+\ln\left(2+d(\Omega_t)\,\delta_{t,\beta}\right)\right).$$

Since $\nabla^2 S$ is smooth, it then follows that

$$\left\|\nabla u \nabla^{\perp} \varphi(t)\right\|_{\beta} \le c(\eta(T))\left\|\nabla^{\perp} \varphi(t)\right\|_{\beta} \cdot\left(\ln\left(2+|\omega|_0^{-1}|\omega(t)|_{\beta}\, d(\Omega_t)\right)+\ln\left(2+d(\Omega_t)\,\delta_{t,\beta}\right)\right),$$

where we have used the assertion of Lemma 4.2. Since $\partial\Phi/\partial t = -\nabla H(\Phi, t)$, from (4.5) and (4.8), it is clear that $d(\Omega_t) \le 2|\Phi|_0 \le c(\eta(T))$. Hence

$$\left\|\nabla u \nabla^{\perp} \varphi(t)\right\|_{\beta} \le c(\eta(T))\left\|\nabla^{\perp} \varphi(t)\right\|_{\beta} \ln\left(2+\delta_{t,\beta}\right). \tag{4.11}$$

Next, from (4.10), we have for $x \in \Omega_t$,

$$\nabla^{\perp} \varphi(x,t) = \nabla^{\perp} \varphi_0\left(\Phi^{-1}(x,t)\right) + \int_0^t \left(\nabla u \nabla^{\perp} \varphi\right)\left(\Phi\left(\Phi^{-1}(x,t), s\right), s\right) ds + \int_0^t \left((\omega - H)\nabla^{\perp} \varphi\right)\left(\Phi\left(\Phi^{-1}(x,t), s\right), s\right) ds.$$

Therefore, by using Lemma 4.3, we find

$$\begin{aligned}\left|\nabla^{\perp} \varphi(t)\right|_{\beta} \le{}& \left|\nabla^{\perp} \varphi_0\right|_{\beta} \exp\left(\int_0^t |\nabla u(\tau)|_0\, d\tau\right) \\ &+c\int_0^t \left(\left|\nabla u \nabla^{\perp} \varphi(s)\right|_{\beta}\right) \exp\left(\int_s^t |\nabla u(\tau)|_0\, d\tau\right) ds \\ &+c\int_0^t \left(\left|(\omega - H)\nabla^{\perp} \varphi(s)\right|_{\beta}\right) \exp\left(\int_s^t |\nabla u(\tau)|_0\, d\tau\right) ds.\end{aligned} \tag{4.12}$$

It follows from (4.11) and (4.8) that

$$\begin{aligned}\left\|\nabla^{\perp} \varphi(t)\right\|_{\beta} \exp\left(-\int_0^t |\nabla u(\tau)|_0\, d\tau\right) \le{}& \left\|\nabla^{\perp} \varphi_0(t)\right\|_{\beta} \\ &+c(\eta(T))\int_0^t \|\nabla \varphi(s)\|_{\beta} \cdot \ln\left(2+\delta_{t,\beta}\right) \exp\left(-\int_0^s |\nabla u(\tau)|_0\, d\tau\right).\end{aligned} \tag{4.13}$$

Set

$$f(t) = \|\varphi(t)\|_{1+\beta} \exp\left(-\int_0^t |\nabla u(\tau)|_0\, d\tau\right) + 2.$$

Using the second assertion in Lemma 4.3, we obtain

$$\delta_{t,\beta} \leq \frac{|\nabla\varphi(t)|_\beta}{|\nabla\varphi(0)|_{\inf,\partial\Omega_0}} \exp\left(\int_0^t |\nabla u(\tau)|_0 \, d\tau\right).$$

Therefore

$$\ln(2+\delta_{t,\beta}) \leq c \ln f + \int_0^t |\nabla u(\tau)|_0 \, d\tau.$$

It follows from (4.13) that

$$f(t) \leq c + c(\eta(T)) \int_0^t f(s) \left(\ln f(s) + \int_0^s |\nabla u(\tau)|_0 \, d\tau\right) ds.$$

Denote by $h(t)$ the function on the right-hand side of the above inequality:

$$h(t) = c + c(\eta(T)) \int_0^t f(s) \left(\ln f(s) + \int_0^s |\nabla u(\tau)|_0 \, d\tau\right) ds.$$

Then

$$h'(t) \leq c(\eta(T))\, h(t) \left(\ln h(t) + \int_0^t |\nabla u(\tau)|_0 \, d\tau\right).$$

It follows that

$$\frac{d}{dt} \ln h(t) \leq c(\eta(T)) \left(\ln h(t) + \int_0^t |\nabla u(\tau)|_0 \, d\tau\right).$$

By the standard Gronwall inequality, we obtain

$$\begin{aligned} \ln f \leq \ln h(t) &\leq c e^{c(\eta(T))} \left(1 + \int_0^t c(\eta(T)) \int_0^s |\nabla u(\tau)|_0 \, d\tau \, ds\right) \\ &\leq c e^{c(\eta(T))} \left(1 + \int_0^t c(\eta(T)) \, ds \int_0^t |\nabla u(\tau)|_0 \, d\tau\right) \qquad (4.14) \\ &\leq c(\eta(T)) \left(1 + \int_0^t |\nabla u(\tau)|_0 \, d\tau\right), \end{aligned}$$

where in the last inequality, $c(\eta(T))$ is a constant depending only on $\eta(T)$ and T. By Lemmas 3.4 and 4.2 and equation (4.13) we deduce

$$\int_0^t |\nabla u(s)|_{0,\Omega_s} \, ds = \int_0^t \left|\nabla^2 H(s)\right|_{0,\Omega_s} ds$$

$$\leq c\int_0^t \left(\ln f + \int_0^s |\nabla u(\tau)|_0 \, d\tau\right) ds \tag{4.15}$$
$$\leq c(\eta(T)) + c(\eta(T))\int_0^t \left(\int_0^s |\nabla u(\tau)|_0 \, d\tau\right) ds.$$

It follows that

$$\int_0^t |\nabla u(\tau)|_0 \, d\tau \leq c(\eta(T)), \quad \text{for } t < T.$$

Substituting this into (4.14), we find $\|\nabla\varphi(t)\|_{\beta(T)}$ is bounded uniformly. Therefore, $\delta_{t,\beta}$ is bounded by a constant depending on $\eta(T)$ and T for $t \leq T$. The proof of Lemma 4.4 is complete.

THEOREM 4.5. *Suppose that the assumptions in Lemma* 4.2 *hold. Then there exists a unique* $C^{1+\alpha}(\Omega_0)$ *solution* $\Phi(x,t)$ *of* (2.11) *for* $t < T + T_0$ *for some* $T_0 > 0$ *depending only on* $\eta(T)$.

Proof. By Corollary 3.6, it suffices to show that $\|\nabla\Phi(t)\|_{\alpha,\Omega_0}$ and $\left\|\nabla\Phi^{-1}(t)\right\|_{\alpha,\Omega_t}$ are uniformly bounded by $c(\eta(T))$. Recall that by differentiating (2.3), $\nabla\Phi(x,t)$ satisfies

$$\frac{\partial\nabla\Phi(x,t)}{\partial t} = \nabla u(\Phi(x,t),t)\nabla\Phi(x,t) = -\nabla^2 H(\Phi(x,t),t)\nabla\Phi(x,t).$$

Applying Lemma 3.4 (estimate (3.12)) with $\alpha = \beta(T)$, and using Lemma 4.2 and 4.4, we deduce

$$\left|\nabla^2 H(\Phi(x,t),t)\right| \leq c(\eta(T)).$$

Therefore,

$$\left|\frac{\partial|\nabla\Phi(t)|_{0,\Omega_0}}{\partial t}\right| \leq c(\eta(T))\,|\nabla\Phi(t)|_{0,\Omega_0}.$$

It follows that

$$ce^{-c(\eta(T))} \leq |\nabla\Phi(t)|_{0,\Omega_0} \leq ce^{c(\eta(T))}. \tag{4.16}$$

Since, recalling definitions,

$$\omega(x,t) = \left(J(\Phi)^{-1}\omega_0\right)\left(\Phi^{-1}(x,t)\right), \qquad \varphi(x,t) = \varphi_0\left(\Phi^{-1}(x,t)\right),$$

it follows that

$$\delta_{t,\alpha} + \|\omega(t)\|_{\alpha,\Omega_t} + d(\Omega_t) \leq c(\eta(T))\,\|\Phi(t)\|_{1+\alpha,\Omega_0}.$$

By (3.19), since $A(\Phi) = \Phi$, we obtain

$$\|\Phi(t)\|_{1+\alpha,\Omega_0} \leq c + c(\eta(T))\int_0^t \|\omega(t)\|_{0,\Omega_t}\|\Phi(t)\|_{1+\alpha,\Omega_0}\left(1 + \ln\left(1 + \|\Phi\|_{1+\alpha,\Omega_0}\right)\right) dt.$$

By the Gronwall inequality, it follows that $\|\Phi(t)\|_{1+\alpha,\Omega_0} \leq c(\eta(t))$ for $t < T$. Combining this estimate and (4.16), we also obtain $\left\|\Phi^{-1}(t)\right\|_{1+\alpha,\Omega_0} \leq c(\eta(t))$ for $t < T$. The proof is now complete.

5. Proof of Theorem 2.2. From Theorem 4.5, it suffices to obtain an a priori estimate for $\eta(t)$ defined in (4.4). We shall use the expression (4.1) to estimate $\eta(t)$.

LEMMA 5.1. *Suppose that* Φ *is a* $C^{1+\alpha}$ *solution for* $t<T$, *and* ω *is the vorticity. Then, for* $t<T$,

$$|\omega(t)|_{0,\Omega_s} \le \frac{|\varpi_0|_{0,\Omega_0} e^{\sigma_h(t)}}{1+|\varpi_0|_{0,\Omega_0}\int_0^t e^{\sigma_h(s)}ds}, \tag{5.1}$$

where

$$\sigma_h(t) = \int_0^t |\omega(s)|_{0,\Omega_s}\, ds.$$

Proof. By the maximum principle, we know that $H(x,t) \le |\omega(t)|_{0,\Omega_s}$ for $t<T$. Since $\omega \ge 0$, by (4.3), we obtain that $\omega(\Phi(x,t),t)$ satisfies

$$\frac{d\omega}{dt} = \omega H - \omega^2 \le \omega |\omega(s)|_{0,\Omega_s} - \omega^2.$$

Integrating this differential inequality as in Lemma 4.1, we derive

$$\omega(\Phi(x,t),t) \le \frac{\varpi_0(x)\, e^{\sigma_h(t)}}{1+\varpi_0(x)\int_0^t e^{\sigma_h(s)}ds}.$$

Noticing that the function $x(1+cx)^{-1}$ is increasing in x (for $c>0$), the assertion follows.

Proof of Theorem 2.2. Let σ_h be the integral defined in Lemma 5.1 and

$$f(t) = \int_0^t e^{\sigma_h(s)}ds.$$

Then

$$f'(t) = e^{\sigma_h(t)}, \qquad f''(t) = e^{\sigma_h(t)}\sigma_h'(t) = |\omega(t)|_{0,\Omega_s} f'(t).$$

It follows from (5.1) that

$$f''(t) \le \frac{|\varpi_0|_{0,\Omega_0} (f'(t))^2}{1+|\varpi_0|_{0,\Omega_0} f(t)}$$

or equivalently

$$\frac{f''(t)}{f'(t)} \le \frac{|\varpi_0|_{0,\Omega_0} f'(t)}{1+|\varpi_0|_{0,\Omega_0} f(t)}.$$

By integration, noticing that $f(0)=0,\ f'(0)=1$, we deduce

$$f'(t) \le 1+|\varpi_0|_{0,\Omega_0} f(t). \tag{5.2}$$

Applying the Gronwall inequality, it follows that

$$f(t) \leq \frac{\left(e^{t|\varpi_0|_{0,\Omega_0}} - 1\right)}{|\varpi_0|_{0,\Omega_0}}.$$

Combining this with (5.2), we find

$$\exp\left(\int_0^t |\omega(s)|_{0,\Omega_s}\, ds\right) = f'(t) \leq e^{t|\varpi_0|_{0,\Omega_0}}.$$

Consequently

$$\eta(t) \leq t\,|\varpi_0|_{0,\Omega_0}.$$

The assertion of Theorem 2.2 follows from Theorem 4.5.

By differentiating the equations (1.2) and (1.3) in x, we find that $\nabla\omega$ satisfies a similar system. Applying the same methods, we can also show the following regularity result.

THEOREM 5.2. *Suppose that* $\Omega_0 \in C^{m+1+\alpha}$, $\varpi_0 \in C^{m+\alpha}$. *Then the solution* Φ *of* (2.11) *is in* $C_x^{m+1+\alpha}(\Omega_0)$.

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