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# Unavoidable minors of large 4-connected bicircular matroids

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## Abstract

It is known that any 3-connected matroid that is large enough is certain to contain a minor of a given size belonging one of a few special classes of matroids. This paper proves a similar unavoidable minor result for large 4-connected bicircular matroids. The main result follows from establishing the list of unavoidable minors of large 4-biconnected graphs, which are the graphs representing the 4-connected bicircular matroids. This paper also gives similar results for internally 4-connected and vertically 4-connected bicircular matroids.

*Key words:* bicircular matroid, 4-connected, internally 4-connected, vertically 4-connected, unavoidable minor

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Dedicated to Dr. James G. Oxley on the occasion of his 60th birthday

## 1 Introduction

Our notation and terminology will generally follow [5]. The following result of Ding, Oporowski, Oxley, and Vertigan, from [2], shows that each sufficiently large 3-connected matroid is guaranteed to contain a large minor isomorphic to one of a few types of 3-connected matroids.

**Theorem 1.1** *For every integer  $n$  exceeding two, there is an integer  $N(n)$  such that every 3-connected matroid with at least  $N(n)$  elements has a minor*

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28 *isomorphic to one of  $U_{n,n+2}$ ,  $U_{2,n+2}$ ,  $M(K_{3,n})$ ,  $M^*(K_{3,n})$ ,  $M(\mathcal{W}_n)$ ,  $\mathcal{W}^n$ , or a*  
29 *uniform  $n$ -spike.*

30 Evidently, corollaries for various minor-closed classes of matroids follow by  
31 filtering out the members of the list in Theorem 1.1 that are not in the class  
32 of interest. For instance, we may choose to restrict to graphic matroids.

33 **Corollary 1.2** *For every integer  $n$  exceeding two, there is an integer  $N(n)$*   
34 *such that every simple, 3-connected graph having at least  $N(n)$  edges has a*  
35 *minor isomorphic to one of  $K_{3,n}$  or  $\mathcal{W}_n$ .*

36 The following result of Oporowski, Oxley, and Thomas, from [4], is a stronger  
37 version of Corollary 1.2. Refer to Figure 1 for an illustration of  $V_k$ , which can  
38 be formed by contracting a pair of consecutive rungs of the circular  $k$ -ladder  
39 and simplifying the resulting graph.

40 **Theorem 1.3** *For every integer  $k \geq 3$ , there is an integer  $N$  such that every*  
41 *3-connected graph with at least  $N$  vertices contains a subgraph isomorphic to*  
42 *a subdivision of one of  $\mathcal{W}_k$ ,  $V_k$ , and  $K_{3,k}$ .*

43 The focus of this paper is an unavoidable minor result for bicircular matroids.  
44 As noted above, a result of this type for 3-connected bicircular matroids is  
45 merely a corollary of Theorem 1.1. However, a 4-connected analog of Theo-  
46 rem 1.1 is not known. The following theorem is the main result of this paper.  
47 Here,  $\mathcal{W}_n^2$  can be constructed from the  $n$ -spoked wheel by adding an edge in  
48 parallel to each spoke. The graph  $K_{3,n}^+$  is formed by adding a loop at each  
49 of the  $n$  degree-3 vertices of  $K_{3,n}$ . Finally,  $K_{3,n}^2$  is constructed from  $K_{3,n}$   
50 by adding an edge in parallel to each of the edges incident with a single degree- $n$   
51 vertex.

52 **Theorem 1.4** *For every integer  $n$  exceeding four, there is an integer  $N(n)$*   
53 *such that every 4-connected bicircular matroid with at least  $N(n)$  elements has*  
54 *a minor isomorphic to one of  $B(\mathcal{W}_n^2)$ ,  $B(K_{3,n}^+)$ , or  $B(K_{3,n}^2)$ .*

55 The proof of this result makes use of a type of graph connectivity called *bicon-*  
56 *nectivity*. Section 2 provides an equivalent characterization of  $n$ -biconnectivity  
57 that is used in Section 4 to prove Theorem 1.4.

58 In Section 3 we analyze the graphic structure of size- $n$  cocircuits in  $n$ -connected  
59 bicircular matroids. This is used in Section 5 to prove the following internally  
60 4-connected analog of Theorem 1.4.

61 **Theorem 1.5** *For every integer  $n$  exceeding four, there is an integer  $N'(n)$*   
62 *such that every internally 4-connected bicircular matroid with at least  $N'(n)$*   
63 *elements has a minor isomorphic to  $B(\mathcal{W}_n)$  or  $B(K_{3,n})$ .*

64 Finally, we prove a vertically 4-connected version of the main result in Sec-  
65 tion 6. Recall that, by definition, a vertically 4-connected may not be 3-  
66 connected. For simplicity, we assume in the next result the matroids under  
67 consideration are 3-connected.

68 **Theorem 1.6** *For each integer  $n$  exceeding four, there is an integer  $N''(n)$   
69 such that every vertically 4-connected and 3-connected bicircular matroid on  
70 at least  $N''(n)$  elements has a restriction isomorphic to  $U_{2,n}$ , or a minor iso-  
71 morphic to one of  $B(\mathcal{W}_n^2)$ ,  $B(K_{3,n}^+)$ , or  $B(K_{3,n}^2)$ .*

## 72 2 Preliminaries

73 Let  $G$  be a graph. The bicircular matroid of  $G$ , denoted by  $B(G)$ , is the matroid  
74 with ground set  $E(G)$ , and a subset of  $E(G)$  is a circuit if it is the edge set  
75 of a minimal connected subgraph of  $G$  that contains at least two cycles. A  
76 subgraph of  $G$  is called a  $\Theta$ -graph if it consists of two distinct vertices and  
77 three internally disjoint paths connecting them; a subgraph is called a *tight*  
78 *handcuff* if it consists of two cycles having just one vertex in common; and a  
79 subgraph is called a *loose handcuff* if it consists of two disjoint cycles and a  
80 minimal connecting path. It is easy to see that a circuit of  $B(G)$  is either a  
81  $\Theta$ -graph, a tight handcuff, or a loose handcuff, shown in Figure 3. A subgraph  
82 of  $G$  is called a *bicycle* if it is a  $\Theta$ -graph, a tight handcuff, or a loose handcuff.

83 Wagner defines  $n$ -biconnectivity in [7] with respect to  $k$ -biseparations as fol-  
84 lows.

Let  $(E_1, E_2)$  partition the edge set  $E$  of a connected graph  $G = (V, E)$ . For  
 $i \in \{1, 2\}$ , let  $G_i$  denote the subgraph of  $G$  induced by  $E_i$ . We say  $(E_1, E_2)$  is  
a  $k$ -biseparation of  $G$ , for  $k \geq 1$ , if each of  $|E_1|$  and  $|E_2|$  is at least  $k$ , and

$$|V(G_1) \cap V(G_2)| = \begin{cases} k - 1 & \text{if neither } G_1 \text{ nor } G_2 \text{ is acyclic} \\ k & \text{if exactly one or all three of } G_1, G_2, \text{ and } G \text{ are acyclic} \\ k + 1 & \text{if both } G_1 \text{ and } G_2 \text{ are acyclic, but } G \text{ is not acyclic} \end{cases}$$

85 For  $n$  a positive integer, a graph is  $n$ -biconnected if it has no  $k$ -biseparation  
86 for  $k < n$ .

87 The next theorem of Wagner from [7] shows that biconnectivity is the version  
88 of graphic connectivity corresponding to matroid connectivity in bicircular  
89 matroids.

90 **Theorem 2.1** *Let  $G$  be a connected graph. Then  $B(G)$  is  $n$ -connected if and  
91 only if  $G$  is  $n$ -biconnected.*

92 Here we give an equivalent characterization for  $n$ -biconnectivity.

93 **Lemma 2.2** *For  $n \geq 3$ , a graph  $G$  on at least  $n$  vertices and at least  $2n - 2$*   
94 *edges is  $n$ -biconnected if and only if each of the following holds:*

- 95 (1)  $G$  has no vertex cut of size at most  $n - 2$ .  
96 (2)  $\delta(G)$ , the minimum degree of  $G$ , is at least  $n$   
97 (3)  $G$  has no bicycle of size at most  $n - 1$

98 *Proof.*

99 Equivalence holds for  $n = 3$  by Wagner in [7]. Suppose that  $G = (V, E)$  is  
100  $n$ -biconnected for a fixed  $n > 3$  and that the lemma holds for smaller values  
101 of  $n$ . Since  $G$  is  $(n - 1)$ -biconnected,  $\delta(G) \geq n - 1$ , and  $G$  has no vertex  
102 cut of size less than  $n - 2$ . Suppose  $G$  has a vertex cut  $W$  of size  $n - 2$ .  
103 Let  $H$  be a component of  $G - W$ . Let  $E_1$  denote the edges of  $G$  having at  
104 least one end in  $V(H)$ . Let  $E_W$  denote the edges of  $G$  having both ends in  
105  $W$ . Let  $E_2 = E - E_1 \cup E_W$ . By  $\delta(G) \geq n - 1$  and the minimality of the  
106 vertex cut  $W$ , we have that each of  $|E_1|$  and  $|E_2|$  is at least  $n - 1$ . Since  $G$   
107 is  $n$ -biconnected, we have that  $(E_i, E_j \cup E_W)$  is not an  $(n - 1)$ -biseparation  
108 for  $(i, j) \in \{(1, 2), (2, 1)\}$ . Up to relabeling, we have that the subgraph  $G_1$  of  
109  $G$  induced by  $E_1$  is acyclic. Since  $\delta(G) \geq n - 1$  we have that a leaf vertex in  
110  $G_1 - W$  must be adjacent to all  $n - 2$  vertices of  $W$ . By acyclicity, there can  
111 be no such vertex. This contradicts that  $W$  is a vertex cut.

112 Suppose  $G$  has a vertex  $v$  of degree  $n - 1$ . Since  $G$  has no vertex cut of size  
113 at most  $n - 2$ , the subgraph induced by the edges incident with  $v$  is acyclic.  
114 Thus  $G - v$  is acyclic since  $G$  has no  $(n - 1)$ -biseparation. Each leaf vertex  
115 of  $G - v - N(v)$  is adjacent to at least  $\delta(G) - 1 \geq n - 2$  members of  $N(v)$ ,  
116 where  $N(v)$  denotes the set of neighbors of  $v$ . Since  $G - v - N(v)$  is acyclic,  
117 each connected component of  $G - v - N(v)$  consists of exactly one vertex.  
118 Since  $\delta(G) \geq n - 1$ , every such vertex must be adjacent to all vertices of  $N(v)$ .  
119 Therefore,  $G - v - N(v)$  consists of exactly one vertex of degree  $n - 1$ , so  $G$   
120 is isomorphic to  $K_{2, n-1}$ , a contradiction to  $\delta(G) > 2$ .

121 By the inductive assumption,  $G$  has no bicycle of size less than  $n - 1$ . Suppose  
122  $G$  has a bicycle of size  $n - 1$  with edge set  $E_1$ . Let  $E_2 = E - E_1$ . Then  
123  $|E_2| \geq 2n - 2 - (n - 1) = n - 1$ , and  $|V(G_1) \cap V(G_2)| = |V(G_1)| = n - 2$ . Since  
124  $G$  has no  $(n - 1)$ -biseparation,  $G_2$  must be acyclic. However,  $G_2$  has at least  
125  $n - (n - 2) = 2$  vertices and therefore at least two leaf vertices; every such leaf  
126 vertex is adjacent to all members of  $V(G_1)$ , a contradiction to acyclicity.

127 Now suppose  $G = (V, E)$  is a graph satisfying the three conditions in the  
128 statement of the lemma for some  $n > 3$  and that the equivalence holds for  
129 smaller values of  $n$ . By assumption,  $G$  has no  $k$ -biseparation for  $k < n - 1$ .  
130 Suppose  $G_1$  and  $G_2$  are induced by an  $(n - 1)$ -biseparation  $(E_1, E_2)$ . First,

131 suppose that  $|V(G_1) \cap V(G_2)| = n - 2$ . Since  $G$  has no size- $(n - 2)$  cutset, at  
 132 least one of  $V(G_1) - V(G_2)$  and  $V(G_2) - V(G_1)$  is empty – assume the former.  
 133 Then  $|E_1| \geq n - 1$  and  $|V_1| = n - 2$ , so  $G_1$  contains a bicycle of size at most  
 134  $n - 1$ , a contradiction. Hence  $|V(G_1) \cap V(G_2)| \geq n - 1$ .

135 Next suppose that  $|V(G_1) \cap V(G_2)| = n - 1$ . The graph  $G$  is not acyclic  
 136 by assumption, so we may assume  $G_1$  is acyclic. Since  $|E_1| \geq n - 1$  and  
 137  $|V(G_1) \cap V(G_2)| = n - 1$ , it follows that  $V(G_1) - V(G_2) \neq \emptyset$ . Since  $\delta(G) \geq n$ ,  
 138 a leaf vertex of  $V(G_1) - V(G_2)$  is adjacent to all vertices of  $V(G_1) \cap V(G_2)$ .  
 139 As  $G_1$  is acyclic, there is only one such vertex. This contradicts the fact that  
 140  $\delta(G) \geq n$ .

141 Therefore, we may assume that  $|V(G_1) \cap V(G_2)| = n$ , so both  $G_1$  and  $G_2$   
 142 are acyclic. First we show that one of  $V(G_1) - V(G_2)$  and  $V(G_2) - V(G_1)$   
 143 is empty. Suppose that neither  $V(G_1) - V(G_2)$  nor  $V(G_2) - V(G_1)$  is empty.  
 144 Since each of  $G_1$  and  $G_2$  is acyclic and  $\delta(G) \geq n$ , each of  $V(G_1) - V(G_2)$  and  
 145  $V(G_2) - V(G_1)$  must have only one vertex by the pigeonhole principle. So  $G$   
 146 is isomorphic to  $K_{2,n}$ , a contradiction.

147 Therefore we may assume that  $V(G_1) - V(G_2) = \emptyset$ . Then  $|E_1| = n - 1$ . Thus  
 148  $|V(G_2) - V(G_1)| \in \{0, 1\}$ . If  $V(G_2) - V(G_1) \neq \emptyset$  then a leaf of  $G_1$  has degree  
 149 2 in  $G$ , a contradiction. Therefore  $V(G_2) - V(G_1) = \emptyset$ . Hence  $G$  is a graph  
 150 on  $2n - 2$  edges and  $n$  vertices. The sum of the degrees of vertices of  $G$  is at  
 151 least  $n\delta(G) \geq 4n$ . However,  $2|E| = 4n - 4$ , a contradiction. Thus,  $G$  has no  
 152  $(n - 1)$ -biseperation, so  $G$  is  $n$ -biconnected.  $\square$

### 153 3 The graphic structure of small cocircuits in $n$ -connected bicircu- 154 lar matroids

155 The following from [3] is Matthews's description of a hyperplane of  $B(G)$  in  
 156 the underlying graph  $G$ , which we assume to be connected and containing a  
 157 bicycle. A hyperplane  $H$  is a collection of edges of  $G$  such that the subgraph  
 158 with vertex set  $V(G)$  and edge set  $H$  consists of

- 159 (1) exactly one acyclic component  $H_0$ , which may be an isolated vertex; and
- 160 (2) a collection of other components, each of which is cyclic;

161 such that all edges of  $E(G) \setminus H$  have at least one endpoint in  $H_0$ .

162 Evidently, a cocircuit of  $B(G)$  is a minimal set of edges  $X$  such that  $G - X$   
 163 has exactly one acyclic component. In general, the edges of a cocircuit need  
 164 not form a bond in  $G$  as they would in the case of graphic matroids. The  
 165 results below describe small cocircuits in the underlying graphs of  $n$ -connected

166 bicircular matroids. Before exploring this graphic structure, we consider the  
 167 following trivial consequence of the minimum degree condition in Lemma 2.2  
 168 that will be used frequently in our description of these small cocircuits.

**Lemma 3.1** *Let  $G$  be a connected graph. Suppose  $B(G)$  is  $n$ -connected, for some  $n \geq 3$ . Let  $X$  be a cocircuit of  $B(G)$ . Let  $H_0$  denote the unique acyclic component of  $G - X$ . Then*

$$2|X| \geq \sum_{\substack{v \in V(H_0); \\ d_{G-X}(v) < n}} n - d_{G-X}(v)$$

169 Recall that a triangle is a 3-element circuit and a triad is a 3-element cocircuit.  
 170 We now consider triads in 3-connected bicircular matroids.

171 **Lemma 3.2** *Let  $G$  be a connected graph having at least seven edges. Suppose*  
 172  *$B(G)$  is 3-connected. If  $X \subseteq E(G)$  is a triad of  $B(G)$ , then the edges of  $X$*   
 173 *are all incident with a common vertex; or  $G|X$  is isomorphic to  $P_4$ , and the*  
 174 *set of edges incident to either of the two internal vertices of this path consists*  
 175 *of the edges of  $X$  along with a single edge in parallel to the middle edge of the*  
 176 *path.*

177 *Proof.* We have that  $G - X$  contains exactly one acyclic component  $H_0$ . Evi-  
 178 dently  $G - X$  has at most one cyclic component  $H_1$  since  $G$  is 2-connected by  
 179 Lemma 2.2. If  $H_0$  has exactly one vertex, we are done. Assume  $H_0$  is a tree  
 180 containing at least two vertices. Thus,  $H_0$  has at least two leaf vertices. By  
 181 Lemma 3.1,  $H_0$  has at most three leaf vertices.

182 If all edges of  $X$  have both ends in  $H_0$ , then  $H_0$  is a tree and  $|E(H_0)| =$   
 183  $|E(G)| - 3 \geq 7 - 3 = 4$ . Since  $H_0$  has at most three leaves, it is easy to see  
 184 that either  $H_0$  is a path of length at least 4, or  $H_0$  has exactly three leaves and  
 185 at least one degree-2 vertex. However, each of these contradicts Lemma 3.1.

186 So we may assume that an edge of  $X$  has one end in  $H_0$  and one end in a  
 187 cyclic component  $H_1$  of  $G - X$ . Since  $G$  is 2-connected, there is at least one  
 188 other  $H_0$ - $H_1$  edge of  $X$ . Therefore,  $H_0$  has exactly two leaf vertices, say  $u$  and  
 189  $v$ , and these are the only vertices in  $H_0$ . Each is incident with an  $H_0$ - $H_1$  edge  
 190 of  $X$ . Since  $\delta(G) \geq 3$ , the third edge of  $X$  must be incident to both  $u$  and  $v$ .

191 □

192 A similar proof technique establishes the graphic structure of  $n$ -cocircuits in  
 193  $n$ -connected bicircular matroids for  $n \geq 4$ .

194 **Lemma 3.3** *Suppose  $G$  is a connected graph having at least seven edges, and*  
 195  *$B(G)$  is  $n$ -connected for some  $n \geq 4$ . If  $X \subseteq E(G)$  is a size- $n$  cocircuit of*

196  $B(G)$ , then the edges of  $X$  are all incident with a common vertex.

197 *Proof.* As in the proof of Lemma 3.2, we may assume that  $H_0$  has at least two  
198 vertices. Since  $2n < 3(n-1)$ ,  $H_0$  has exactly two leaf vertices by Lemma 3.1,  
199 so  $H_0$  is a path. Furthermore,  $2n < 2(n-1) + 2(n-2)$  so  $H_0$  is  $P_2$  or  $P_3$ .

200 First suppose that all edges of  $X$  have both ends in  $H_0$ . So  $|V(G)| = 2$  or  $3$ ,  
201 and  $|E(G)| \geq 7$ . It is easy to see that  $G$  must contain a bicycle of size at most  
202  $3$ , contradicting the  $n$ -biconnectivity.

203 Thus there is an edge in  $X$  that has an end in a cyclic component  $H_1$  of  $G-X$ .  
204 By the  $(n-1)$ -connectivity of  $G$ , there are least 2 such edges. Then there are  
205 at most  $2n-2$  ends of the edges of  $X$  in  $H_0$ . Thus  $H_0$  is  $P_2$ . Since bicycles of  
206  $G$  must have at least four edges, at most one edge of  $X$  has both ends in  $H_0$ .  
207 Then there are at most  $n-1+2 = n+1$  ends of the edges of  $X$  in  $H_0$ . Since  
208  $n+1 < 2n-2$ , this is a contradiction.  $\square$

## 209 4 Unavoidable minors of 4-connected bicircular matroids

210 Before proving the main result of the paper, we recall that if a graph  $H$  is  
211 a minor of a graph  $G$ , then the bicircular matroid  $B(H)$  is a minor of  $B(G)$   
212 [8]. The next result can be found in Biedl [1]; one may prove it by a simple  
213 counting argument.

214 **Lemma 4.1** *A maximal matching in a max-deg- $k$  graph with  $m$  edges has size*  
215 *at least  $\frac{m}{2k-1}$ .*

216 The next lemma is the main result of this section.

217 **Lemma 4.2** *For each  $n$  there is an  $R(n)$  such that every 3-connected graph*  
218 *on at least  $R(n)$  vertices having minimum degree at least four has a minor*  
219 *isomorphic to one of  $W_n^2$ ,  $K_{3,n}^+$ , or  $K_{3,n}^2$ .*

220 *Proof.* By Theorem 1.3, there is an  $R$  such that each 3-connected graph on  
221 at least  $R$  vertices has a subgraph isomorphic to a subdivision of  $W_k$ ,  $K_{3,k}$ ,  
222 or  $V_k$  for  $k = 4n^2 - 2n - 4$ . Suppose  $G$  is a 3-connected graph on at least  $R$   
223 vertices. Since  $k = 4n^2 - 2n - 4 > 4n$ , if  $G$  has a  $W_k$ - or  $V_k$ -subdivision as  
224 a subgraph, then  $G$  has a  $W_n^2$ -minor, and we are done. Assume then that  $G$   
225 has a  $K_{3,k}$ -subdivision as a subgraph. That is,  $G$  has vertices  $u_1, u_2, u_3, v_1,$   
226  $v_2, \dots, v_k$  such that there for each  $i \in \{1, 2, \dots, k\}$  there are paths  $P_{i,1}, P_{i,2},$   
227 and  $P_{i,3}$  from  $v_i$  to  $u_1, u_2,$  and  $u_3$ , respectively, such that  $P_{i_1, j_1}$  and  $P_{i_2, j_2}$  are  
228 internally vertex-disjoint whenever  $(i_1, j_1) \neq (i_2, j_2)$ .



229 Let  $e \in E(G)$ . Note that if  $e$  satisfies either of the following conditions, then  
 230  $G/e$  contains a  $K_{3,k}$  subdivision having small and large sides  $\{u_1, u_2, u_3\}$  and  
 231  $\{v_1, v_2, \dots, v_k\}$ , respectively, such that  $d_{G/e}(v_j) \geq 4$  for each  $j \in \{1, 2, \dots, k\}$ .

- 232 (1) For some  $a \in \{1, 2, 3\}$  and  $b \in \{1, 2, \dots, k\}$ ,  $e$  is an edge on the path  $P_{a,b}$   
 233 that is incident with  $u_a$  but has its other end in  $V(G) - \{v_1, v_2, \dots, v_k\}$ .  
 234 (2) Each path  $P_{i,j}$  has length one and  $e$  is an edge of  $G$  with one end in  
 235  $\{v_1, v_2, \dots, v_k\}$  and the other end in  $V(G) - \{u_1, u_2, u_3, v_1, v_2, \dots, v_k\}$ .

236 Obtain a minor  $H$  of  $G$  by consecutively contracting edges of the types given  
 237 above until no such edges remain, followed by deleting all edges not incident  
 238 with some  $\{v_1, v_2, \dots, v_k\}$ .

239 Now,  $H$  consists of a  $K_{3,k}$ -subgraph with some extra edges added incident  
 240 with the vertices on the large side of the bipartition. By construction, no step  
 241 of the algorithm above decreases the degree of a vertex in  $\{v_1, v_2, \dots, v_k\}$ .  
 242 Hence, each of the  $k = 4n^2 - 2n - 4$  vertices is incident with at least one  
 243 such extra edge. If at least  $n$  of these vertices have adjacent loops, then  $H$  has  
 244 a  $K_{3,n}^+$ -minor. If at least  $3n - 2$  of these vertices are adjacent to a vertex in  
 245  $\{u_1, u_2, u_3\}$  by an edge not in the  $K_{3,k}$ -graph, then at least  $n$  are adjacent to  
 246 the same vertex by the pigeonhole principle, so  $H$  has a  $K_{3,n}^2$  minor. Assume  
 247 neither of these cases occurs. Let  $E_1$  be the set of non-loop edges of  $H$  that  
 248 have both ends in  $\{v_1, v_2, \dots, v_k\}$ , let  $H_1 = \text{span}_H(E_1)$  and let  $Z = V(H_1)$ .  
 249 Then  $|Z| \geq (4n^2 - 2n - 4) - (n - 1) - (3n - 3) = 4n^2 - 6n$  and every vertex  
 250 in  $Z$  is adjacent to some other vertex in  $Z$ . We have that  $H_1$  has at least  
 251  $\frac{|Z|}{2} \geq 2n^2 - 3n$  edges. If some vertex  $v_i \in Z$  has degree greater than  $n - 1$  in  
 252  $H_1$ , then  $H$  has  $K_{3,n}^2$ -minor by contraction of the edge  $v_i u_1$ . Assume then that  
 253 the maximum degree in  $H_1$  is at most  $n - 1$ . Then by Lemma 4.1,  $H_1$  has a  
 254 matching of size at least  $\frac{2n^2 - 3n}{2n - 3} = n$ . Thus  $H$  has a  $K_{3,n}^2$ -minor by contraction  
 255 of each edge in this matching.  $\square$

256 **Corollary 4.3** *For each  $n$  there is an  $N(n)$  such that every 4-biconnected*  
 257 *graph on at least  $N(n)$  edges has a minor isomorphic to one of  $W_n^2$ ,  $K_{3,n}^+$ , or*  
 258  *$K_{3,n}^2$ .*

259 *Proof.* Note that a 4-biconnected graph  $G$  contains at most one loop at each  
 260 vertex, and each parallel class of edges has size at most two. Therefore,  
 261  $|E(G)| \leq |V(G)| + 2 \binom{|V(G)|}{2} = |V(G)|^2$ . Hence  $|V(G)| \geq \sqrt{|E(G)|}$ . Fix  $n$ . Let  
 262  $R(n)$  be given as in Lemma 4.2. If  $|E(G)| \geq R(n)^2$  then  $G$  is a 3-connected  
 263 graph with  $\delta(G) \geq 4$  on at least  $R(n)$  vertices, so  $G$  has one of the given  
 264 minors.  $\square$

265 It is a trivial matter to prove Theorem 1.4 from the above corollary.

266 *Proof.* [Proof of Theorem 1.4]The theorem follows from Corollary 4.3 since a  
 267 sufficiently large 4-connected bicircular matroid can be represented by a large  
 268 4-biconnected graph, which in turn must have one of the given large minors.  
 269  $\square$

## 270 5 Unavoidable minors of internally 4-connected bicircular matroids

271 Recall that a matroid  $M$  is internally 4-connected if  $M$  is 3-connected and  
 272 for every 3-separation  $(X, Y)$  of  $M$ , either  $|X| = 3$  or  $|Y| = 3$ . It is clear  
 273 that a triangle in a bicircular matroid  $B(G)$  is a set of three parallel edges, a  
 274 set of two parallel edges and a loop at one end, or two loops at two distinct  
 275 vertices and an edge between them in the associated graph  $G$ . Lemma 3.2  
 276 describes what a triad looks like in a 3-connected bicircular matroid. Note  
 277 that the exceptional case in Lemma 3.2 gives rise to a 3-separating set of size  
 278 4, thus does not occur in an internally 4-connected bicircular matroid  $B(G)$   
 279 when  $|E(G)| \geq 8$ . Therefore, every triad in an internally 4-connected bicircular  
 280 matroid corresponds to either a degree-3 vertex, or a degree-4 vertex incident  
 281 to exactly one loop in the underlying graph.

282 By Lemma 2.2, the graph underlying an internally 4-connected bicircular ma-  
 283 troid is 2-connected and has a minimum degree of at least three. However, us-  
 284 ing Wagner’s original definition of biconnectivity, we see that the 2-separations  
 285 in such a graph are highly restricted.

286 **Lemma 5.1** *Let  $G$  be a connected graph having at least six edges. If  $B(G)$  is*  
 287 *internally 4-connected and  $G$  has a 2-vertex cut, then one side of the separation*  
 288 *consists of a single vertex having exactly three incident edges.*

289 *Proof.* Since  $\delta(G) \geq 3$ , each side of the 2-separation is cyclic. Therefore, the  
 290 2-vertex cut in  $G$  naturally induces a “small” 3-biseparation  $(E_1, E_2)$  in  $G$ .  
 291 Assume  $|E_1| = 3$  since  $G$  is internally 4-connected. Thus  $|V(G_1) - V(G_2)| = 1$ .  
 292  $\square$

293 Each 2-separation in the graph underlying an internally 4-connected bicircular  
 294 matroid must have one of the configurations given in Figure 4.

295 Now it is easy to see that we have the following graphic characterization for  
 296 a bicircular matroid to be internally 4-connected.

297 **Lemma 5.2** *Let  $G$  be a connected graph having at least eight edges. Then*  
 298  *$B(G)$  is internally 4-connected if and only if each of the following holds.*

- 299 (1)  $G$  is 2-connected.
- 300 (2) There exists at most one loop at each vertex.

- 301 (3)  $\delta(G)$ , the minimum degree of  $G$ , is at least 3  
302 (4) Every vertex cut of size 2 must have one of the forms shown in Figure 4;  
303 moreover, there exists no edges between and no loops at the two cut ver-  
304 tices.  
305 (5) Every parallel class of edges has size at most 3.  
306 (6) For each parallel class of size 3, there exists no loop at either end.  
307 (7) For each parallel class of size 2, there exists at most one loop at the two  
308 ends.

309 We now prove our result on the unavoidable minors of large internally 4-  
310 connected bicircular matroids.

311 *Proof.*[Proof of Theorem 1.5] First note that the matroids  $B(\mathcal{W}_n)$  and  $B(K_{3,n})$   
312 are internally 4-connected by Lemma 5.2.

313 Suppose  $G$  is a connected graph for which  $B(G)$  is internally 4-connected. A  
314 parallel class of edges in  $G$  has size at most three, and there is at most one  
315 loop at each vertex. Therefore,  $|E(G)| \leq |V(G)| + 3\binom{|V(G)|}{2} \leq \frac{3}{2}|V(G)|^2$ . Thus  
316  $|V(G)| \geq \sqrt{\frac{2}{3}|E(G)|}$ .

317 Now suppose  $G$  is a connected graph underlying an internally 4-connected  
318 bicircular matroid  $B(G)$  having at least  $\frac{3}{2}R^4$  elements in its ground set, where  
319  $R$  is an integer for which any 3-connected graph on at least  $R$  vertices has a  
320 minor isomorphic to  $\mathcal{W}_n$  or  $K_{3,n}$  as given by Corollary 1.2.

If  $G$  has a 2-separation, we have by Lemma 5.1 that one side of the separation consists of a single degree-3 vertex that is adjacent to exactly two vertices, namely the two cut vertices. Call such a degree-3 vertex a *tick*. A vertex that is not a tick is a *non-tick*. There is a natural injection between the set of ticks and the set of pairs of non-ticks given by matching a tick with its associated pair of 2-separating non-tick vertices. Let  $\tau$  denote the number of ticks in  $G$ , and let  $\eta$  denote the number of non-tick vertices. We have that  $\tau \leq \binom{\eta}{2}$  and  $\eta + \tau = |V(G)|$ . By  $\eta \geq 1$  we have  $\frac{\eta-1}{2} + 1 \leq \eta$ , so

$$\eta^2 \geq \eta \left( \frac{\eta-1}{2} + 1 \right) = \binom{\eta}{2} + \eta \geq \tau + \eta = |V(G)|$$

321 Note that the graph resulting from the contraction of a link edge incident  
322 with a tick is still 2-connected. Furthermore, any 2-separations of the resultant  
323 graph are also (up to identification of vertices via contraction) 2-separations  
324 of  $G$ . Thus, we can consecutively contract link edges incident with ticks to  
325 obtain a 3-connected graph  $H$  having  $\eta \geq \sqrt{|V(G)|}$  vertices.

326 Recall that  $G$  has at least  $\frac{3}{2}R^4$  edges, so  $G$  has at least  $R^2$  vertices. Hence,

327  $G$  has a 3-connected minor having at least  $R$  vertices. Thus,  $G$  has a minor  
 328 isomorphic to one of  $\mathcal{W}_n$  or  $K_{3,n}$ , so  $B(G)$  has a minor isomorphic to one of  
 329  $B(\mathcal{W}_n)$  or  $B(K_{3,n})$ .  $\square$

## 330 6 Bicircular matroid that are vertically 4-connected and 3-connected

331 In this section we study bicircular matroids that are both vertically 4-connected  
 332 and 3-connected. Since a rank-2 flat in a 3-connected bicircular matroid is a  
 333 class of parallel non-loop edges plus the set of loops at the two end vertices,  
 334 the next result follows easily from Lemma 2.2.

335 **Lemma 6.1** *If  $G$  is a connected graph on at least four vertices such that  $B(G)$   
 336 is vertically 4-connected and 3-connected, then  $G$  is 3-connected and  $\delta(G) \geq 4$ .*

337 Now we are ready to prove Theorem 1.6.

338 *Proof.*[Proof of Theorem 1.6] Suppose  $G$  is a connected graph such that  $B(G)$  is  
 339 3-connected and vertically 4-connected and  $|E(G)| \geq N'' = \frac{n-1}{2}R(n)^2$ , where  
 340  $R(n)$  is given as in Lemma 4.2.

341 If  $G$  has a parallel class of edges of size at least  $n$ , then  $B(G)$  has a  $U_{2,n}$ -  
 342 restriction. So we may assume that each parallel class of edges has size at  
 343 most  $n-1$ . Since  $B(G)$  is 3-connected,  $G$  has at most one loop at each vertex.  
 344 Therefore we have  $|E(G)| \leq |V(G)| + (n-1) \binom{|V(G)|}{2} = \frac{n-1}{2}|V(G)|^2 - \frac{n-3}{2}|V(G)|$ .  
 345 Since  $n \geq 4$ ,  $|E(G)| \leq \frac{n-1}{2}|V(G)|^2$ . Therefore,  $|V(G)| \geq \sqrt{\frac{2}{n-1}|E(G)|} \geq$   
 346  $\sqrt{\frac{2}{n-1} \cdot \frac{n-1}{2}R(n)^2} = R(n)$ . By Lemma 6.1,  $G$  is a 3-connected graph having  
 347 minimum degree at least four. By Lemma 4.2,  $G$  has a minor isomorphic to  
 348 one of  $\mathcal{W}_n^2$ ,  $K_{3,n}^+$ , or  $K_{3,n}^2$ . Thus,  $G$  has one of these minors, so  $B(G)$  has a  
 349 minor isomorphic to the bicircular matroids of one of these graphs.

350  $\square$

## 351 7 Conclusion

352 The class of 4-connected bicircular matroids is admittedly restrictive. However,  
 353 the techniques in this paper center around the biconnectivity property and do  
 354 not readily extend to more general classes of bias matroids. Slilaty and Qin  
 355 offer a version of Wagner's biconnectivity that is generalized to bias matroids  
 356 in [6]. Evidently, the extra attention that must be paid to balanced cycles  
 357 is the inherent complication in obtaining an analog of Lemma 2.2, which we

358 have relied upon in our proof. An extension to 4-connected signed graphic  
359 matroids might be much more easily obtained and would still have the benefit  
360 of providing the list of unavoidable minors of large 4-connected graphs.

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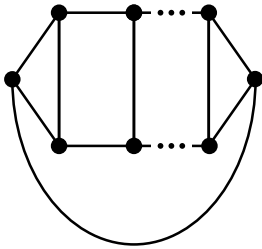


Fig. 1. Illustration of  $V_k$

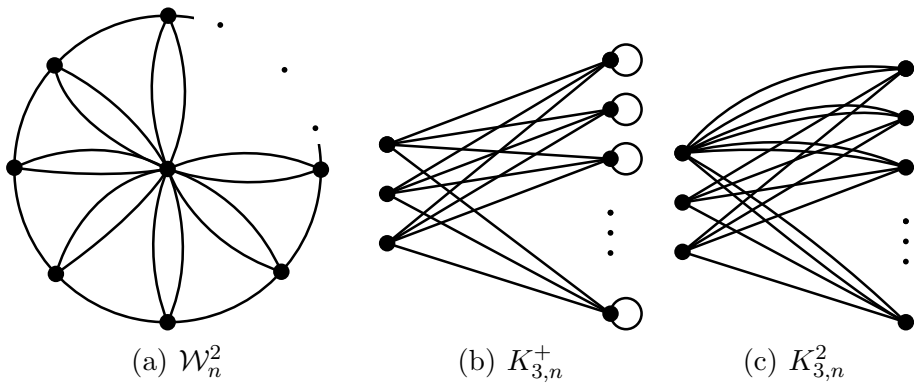


Fig. 2. Unavoidable minors for 4-biconnectivity



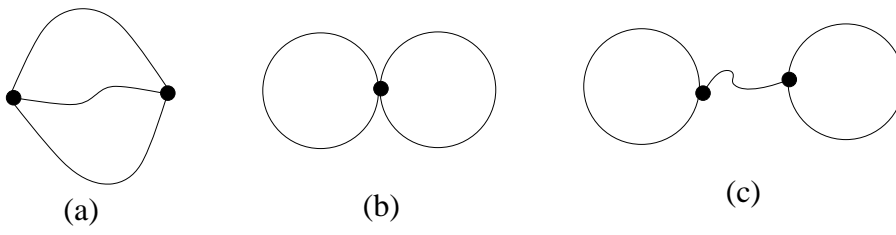


Fig. 3. Three types of bicycles

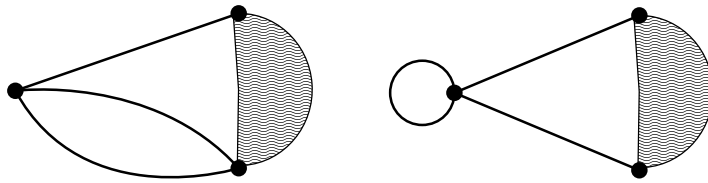


Fig. 4. The 2-separations in graphs underlying internally 4-connected bicircular matroids